REPRESENTATION THEORY OF MANTACI-REUTENAUER ALGEBRAS

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Abstract. We study some aspects of the representation theory of Mantaci-Reutenauer algebras: Cartan matrix, Loewy length, modular representations.

Let W_n be a Coxeter group of type B_n (i.e. the group of permutations σ of $I_n = \{\pm 1, \dots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for every $i \in I_n$) and let R be a commutative ring. Mantaci and Reutenauer [MR] have defined a subalgebra $R\Sigma'(W_n)$ of the group algebra RW_n which contains both the Solomon descent algebra of the symmetric group $\mathfrak{S}_n$ and the one of W_n. In [BH], the authors have provided another construction of the Mantaci-Reutenauer algebra $R\Sigma'(W_n)$ which relies more on the structure of W_n as a Coxeter group. As a consequence of their work, they were able to generalize to this algebra the classical results of Solomon on the Solomon descent algebra (construction of a morphism to the character ring of W_n, description of the radical whenever R is a field of characteristic 0...). For instance, the description of the simple $\mathbb{Q}\Sigma'(W_n)$-modules was obtained in [BH, Proposition 3.11]: they are all of dimension 1.

In this paper, we study the representation theory of $K\Sigma'(W_n)$ whenever $R = K$ is a field of any characteristic: simple modules, radical, projective modules, Cartan matrix... We also define some morphisms between different Mantaci-Reutenauer algebras. Let us gather here some of the main results obtained all along the text:

Theorem. *Let p denote the characteristic of K. Then:*

(a) *There exists a natural morphism of algebras $K\Sigma'(W_n) \to K\Sigma'(W_{n-1})$; it is surjective if $p = 0$.*

(b) *If $p \neq 2$, then the Loewy length of $K\Sigma'(W_n)$ is n. If $p = 2$, then this Loewy length lies in $\{n, n+1, \dots, 2n-1\}$.*

(c) *If p does not divide $|W_n|$ (i.e. is $p = 0$ or $p > \max(2, n)$), then the Cartan matrix of $K\Sigma'(W_n)$ is unitriangular.*

(d) *If p does not divide the order of W_n, then the Cartan matrix of $K\Sigma'(W_n)$ is a submatrix of the Cartan matrix of $K\Sigma'(W_{n+1})$.*

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Note that, in the statement (a), we expect that the homomorphism is surjective even if $p > 0$, but we are unable to prove it. In the statement (b), we expect that the Loewy length of $\mathbb{F}_2\Sigma'(W_n)$ is equal to $2n-1$ whenever $n \geqslant 2$.

The paper is organized as follows. In the first section, we gather (and sometimes improve, or make more precise) some of the principal results of [BH]. In the second section, we study some particular families of left, right and two-sided ideals of $R\Sigma'(W_n)$. In the third section, we introduce a class of *positive* elements of $K\Sigma'(W_n)$ (whenever K is an ordered field) and study the ideals they generate (and also some other properties: centralizer, minimal polynomial). In the fourth section, we study the action of the longest element w_n of W_n on simple modules and on $K\Sigma'(W_n)$: since w_n is central (and is an element of $K\Sigma'(W_n)$), this provides a first decomposition of the Mantaci-Reutenauer algebra (at least when K is not of characteristic 2: we also give a basis of $K\Sigma'(W_n)$ consisting of eigenvectors for the action of w_n by left multiplication). In the fifth section, we define some morphisms between Mantaci-Reutenauer algebras and prove the statement (a) of the Theorem above. In the sixth section, we study the simple modules and compute explicitly the radical of $K\Sigma'(W_n)$ (this is done in any characteristic). Section 7 is devoted to the computation of the Loewy length of $K\Sigma'(W_n)$, that is to the proof of the statement (b) of the above Theorem. We also obtain the Loewy length of the algebra $K \operatorname{Irr} W_n$ in any characteristic. The section 8 is concerned with the projective modules and the Cartan matrix of $K\Sigma'(W_n)$: the statement (c) and (d) of the above Theorem are proved. We also obtain some results about the structure of KW_n as a $K\Sigma'(W_n)$-module. We give in section 9 some numerical results (character tables, primitive idempotents and the Cartan matrices for small values of n). In the final section, we address some questions that are raised by the present work.

Most of this work is largely inspired by works of several authors on Solomon descent algebras (see for instance [A], [BBHT], [APVW], [BP]...). Sections 2, 3, 4, 5 are analogous to [BP, §2, 3, 4] (for §5, see also [A] and [BBHT]). Sections 6 and 8 are inspired by [APVW]). Section 7 is the analogue of [BP, §5]. The question 6 in section 10 has been suggested by a similar question of G. Pfeiffer on Solomon descent algebras.

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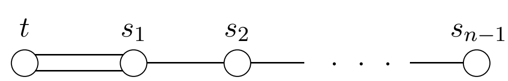
1. NOTATION, PRELIMINARIES

1.A. General notation. All along this paper, R will denote a fixed commutative ring and K a fixed field. If G is a finite group, the group algebra of G over R is denoted by RG and the set of irreducible characters of G over $\mathbb{C}$ is denoted by $\operatorname{Irr} G$. We denote by $R \operatorname{Irr} G$ the ring of formal R-linear combinations of irreducible characters of G (with multiplication given by tensor product). In particular, $\mathbb{Z} \operatorname{Irr} G$ can be identified with the Grothendieck ring of the category of finite dimensional $\mathbb{C}G$-module (which is usually called the *character ring* of G) and $R \operatorname{Irr} G = R \otimes_{\mathbb{Z}} \mathbb{Z} \operatorname{Irr} G$. If A is a finite dimensional K-algebra, we denote by $\operatorname{Rad} A$ its radical.

1.B. Weyl group of type B_n. If $n \geqslant 1$, we denote by (W_n, S_n) a Weyl group of type B_n: write $S_n = \{t, s_1, s_2, \dots, s_{n-1}\}$ in such a way that the Dynkin diagram of W_n is



Let $S_{-n} = \{s_1, s_2, \dots, s_{n-1}\}$ and $W_{-n} =< S_{-n} >$. Note that $W_{-n} \simeq \mathfrak{S}_n$. We denote by $\ell : W_n \to \mathbb{N}$ the length function attached to S_n.

Let $I_n = \{\pm 1, \pm 2, \dots, \pm n\}$. We identify W_n with the group of permutations σ of I_n such that $\sigma(-i) = -\sigma(i)$ for every $i \in I_n$. The identification is as follows: t corresponds to the transposition $(1, -1)$ while s_i corresponds to $(i, i+1)(-i, -i-1)$. Let $t_1 = t$ and, if $1 \leqslant i \leqslant n-1$, let $t_{i+1} = s_i t_i s_i$. As a permutation of I_n, t_i is equal to $(i, -i)$. Now, we set $T_n = \{t_1, \dots, t_n\}$ and $S'_n = S_n \cup T_n$. Then the reflection subgroup $\mathfrak{T}_n$ generated by T_n is naturally identified with $(\mathbb{Z}/2\mathbb{Z})^n$. Therefore $W_n = W_{-n} \ltimes \mathfrak{T}_n$ is, abstractly, the wreath product of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$.

Let $(e_1, \dots, e_n)$ denote the canonical basis of the euclidean $\mathbb{R}$-vector space $\mathbb{R}^n$. If $\alpha \in \mathbb{R}^n$, we denote by s_α the orthogonal reflection such that $s_\alpha(\alpha) = -\alpha$. Let

$$\Phi_n = \{\pm 2e_i \mid 1 \leqslant i \leqslant n\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}.$$

Then Φ_n is a root system and W_n can be also identified with the Weyl group of Φ_n: through this identification, we have $t_i = s_{2e_i}$ and $s_i = s_{e_{i+1}-e_i}$. Let

$$\Delta_n = \{2e_1, e_2 - e_1, e_3 - e_2, \dots, e_n - e_{n-1}\}.$$

Then Δ_n is a basis of Φ_n and we denote by Φ_n^+ the set of roots which are linear combinations with non-negative coefficients of roots in Δ_n. If $\alpha \in \Phi_n$, we write $\alpha > 0$ if $\alpha \in \Phi_n^+$ and $\alpha < 0$ otherwise.

1.C. Signed compositions, bipartitions. A *signed composition* is a finite sequence $C = (c_1, \dots, c_r)$ of non-zero elements of $\mathbb{Z}$. The number r is called the *length* of C and will be denoted by $\lg(C)$. We denote by $\lg^+(C)$ (respectively $\lg^-(C)$) the number of positive (respectively negative) parts of C. In particular, $\lg(C) = \lg^+(C) + \lg^-(C)$. We set $|C| = \sum_{i=1}^r |c_i|$. If $|C| = n$, we say that C is a *signed composition of* n and we write $C \Vdash n$. We also define $C^+ = (|c_1|, \dots, |c_r|) \Vdash n$ and $C^- = -C^+$. We denote by $\mathrm{Comp}(n)$ the set of signed compositions of n. In particular, any composition is a signed composition (any part is positive). Note that

$$|\mathrm{Comp}(n)| = 2.3^{n-1}. \tag{1.1}$$

If $C = (c_1, \dots, c_r)$ and $D = (d_1, \dots, d_s)$ are signed compositions of m and n respectively, we denote by $C \sqcup D$ the signed composition $(c_1, \dots, c_r, d_1, \dots, d_s)$ of $m + n$.

A *bipartition* of n is a pair $\lambda = (\lambda^+, \lambda^-)$ of partitions such that $|\lambda| := |\lambda^+| + |\lambda^-| = n$. We set $\lg^+(\lambda) = \lg(\lambda^+)$, $\lg^-(\lambda) = \lg(\lambda^-)$ and $\lg(\lambda) = \lg(\lambda^+) + \lg(\lambda^-)$. We write $\lambda \Vdash n$ to say that λ is a bipartition of n, and the set of bipartitions of n is denoted by $\mathrm{Bip}(n)$. We define $\hat{\lambda}$ as the signed composition of n obtained by concatenation of λ^+ and $-\lambda^-$. In other words, $\hat{\lambda} = \lambda^+ \sqcup -\lambda^-$. The map $\mathrm{Bip}(n) \to \mathrm{Comp}(n)$, $\lambda \mapsto \hat{\lambda}$ is injective.

Now, let C be a signed composition of n. We define $\boldsymbol{\lambda}(C) = (\lambda^+, \lambda^-)$ as the bipartition of n such that λ^+ (resp. λ^-) is obtained from C by reordering if necessary the positive parts of C (resp. the absolute value of the negative parts of C). Note that $\lg(\boldsymbol{\lambda}(C)) = \lg(C)$, $\lg^+(\boldsymbol{\lambda}(C)) = \lg^+(C)$ and $\lg^-(\boldsymbol{\lambda}(C)) = \lg^-(C)$. One can easily check that the map

$$\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$$

is surjective (indeed, if $\lambda \in \mathrm{Bip}(n)$, then $\boldsymbol{\lambda}(\hat{\lambda}) = \lambda$).

1.D. A class of reflection subgroups of $\boldsymbol{W_n}$. Now, to each $C = (c_1, \dots, c_r) \Vdash n$, we associate a reflection subgroup W_C of W_n which is isomorphic to $W_{c_1} \times \dots \times W_{c_r}$. We proceed as follows: for $1 \le i \le r$, set

$$I_C^{(i)} = \begin{cases} I_{C,+}^{(i)} & \text{if } c_i < 0, \\ I_{C,+}^{(i)} \cup -I_{C,+}^{(i)} & \text{if } c_i > 0, \end{cases}$$

where $I_{C,+}^{(i)}$ is the set of natural numbers k such that $|c_1|+\cdots+|c_{i-1}|+1 \leqslant k \leqslant |c_1|+\cdots+|c_i|$. Then

$$W_C = \{w \in W_n \mid \forall\ 1 \leqslant i \leqslant r,\ w(I_C^{(i)}) = I_C^{(i)}\}$$

is a reflection subgroup generated by

$$S_C = (S_{-n} \cap W_C) \cup \{t_{|c_1|+\cdots+|c_{j-1}|+1} \in T_n \mid c_j > 0\} \qquad \subset S'_n.$$

Note that (W_C, S_C) is a Coxeter group. Moreover, $W_C \simeq W_{c_1} \times \cdots \times W_{c_r}$. Let $T_C = T_n \cap W_C$. Then $W_C = \mathfrak{S}_{C^+} \ltimes < T_C >$, where $\mathfrak{S}_{C^+} = W_{C^-}$.

Now let $\Phi_C = \{\alpha \in \Phi_n \mid s_\alpha \in W_C\}$. Then Φ_C is a root system and W_C is naturally identified with the Weyl group of Φ_C. Let $\Phi_C^+ = \Phi_C \cap \Phi_n^+$ and $\Delta_C = \{\alpha \in \Phi_C^+ \mid s_\alpha \in S_C\}$. Then Δ_C is a basis of Φ_C, and Φ_C^+ is a positive root system of Φ_C.

If $C, D \mathrel{||\!\!=} n$, then we write $C \subset D$ if $W_C \subset W_D$. This defines an order $\subset$ on $\mathrm{Comp}(n)$.

REMARK 1.2 - If $C \subset D$, then $\mathrm{lg}(C) \geqslant \mathrm{lg}(D)$ and $\mathrm{lg}^-(C) \geqslant \mathrm{lg}^-(D)$. If $C \subset D$, $\mathrm{lg}(C) = \mathrm{lg}(D)$ and $\mathrm{lg}^-(C) = \mathrm{lg}^-(D)$, then $C = D$. $\square$

EXAMPLE - It might happen that $C \subset D$ and $\mathrm{lg}^+(C) < \mathrm{lg}^+(D)$. For example, take $C = (-n)$ and $D = (n)$. $\square$

1.E. Conjugacy classes. If $C \mathrel{||\!\!=} n$, we denote by cox_C a Coxeter element of (W_C, S_C). If C, $C' \subset D$ and if W_C and $W_{C'}$ are conjugate under W_D, then we write $C \equiv_D C'$. Note that cox_C and $\mathrm{cox}_{C'}$ are conjugate in W_D if and only if $C \equiv_D C'$. Moreover, every element of W_D is W_D-conjugate to cox_C for some $C \subset D$. If $D = (n)$, we write $\equiv$ instead of $\equiv_D$. We recall the following easy proposition:

Proposition A. *Let $C, D \mathrel{||\!\!=} n$. Then W_C and W_D are conjugate in W_n if and only if $\boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$.*

If $w \in W_n$, we denote by $\boldsymbol{\Lambda}(w)$ the unique bipartition λ of n such that w is conjugate to cox_C for some (every) $C \in \boldsymbol{\lambda}^{-1}(\lambda)$. The map

$$\boldsymbol{\Lambda} : W_n \longrightarrow \mathrm{Bip}(n)$$

is well-defined, surjective and its fibers are precisely the conjugacy classes of W_n: if $\lambda \in \mathrm{Bip}(n)$, we set $\mathcal{C}(\lambda) = \boldsymbol{\Lambda}^{-1}(\lambda)$ and we fix an element $\mathrm{cox}_\lambda \in \mathcal{C}(\lambda)$ (if $C \in \mathrm{Comp}(n)$, $\mathrm{cox}_{\boldsymbol{\lambda}(C)}$ is conjugate to cox_C). We denote by $o(\lambda)$ the order of an element of $\mathcal{C}(\lambda)$: if $\lambda = (\lambda^+, \lambda^-)$ where $\lambda^+ = (\lambda_1^+, \dots, \lambda_k^+)$ and $\lambda^- = (\lambda_1^-, \dots, \lambda_l^-)$, then $o(\lambda)$ is the least common multiple of $\{2\lambda_1^+, \dots, 2\lambda_k^+, \lambda_1^-, \dots, \lambda_l^-\}$.

1.F. Mantaci-Reutenauer algebra. Let $C \mathrel{||\!\!=} n$, then

$$X_C = \{x \in W_n \mid \forall\, w \in W_C,\ \ell(xw) \geqslant \ell(x)\}$$

is a distinguished set of *minimal coset representatives* for W_n/W_C (see [BH, Proposition 2.8 (a)]). It is readily seen that

$$\begin{aligned} X_C &= \{w \in W_n \mid \forall\, s \in S_C,\ \ell(ws) > \ell(w)\} \\ &= \{w \in W_n \mid \forall\, \alpha \in \Phi_C^+,\ w(\alpha) > 0\} \\ &= \{w \in W_n \mid \forall\, \alpha \in \Delta_C,\ w(\alpha) > 0\}. \end{aligned}$$

Now, we set

$$x_C = \sum_{w \in X_C} w \qquad \in RW_n.$$

(Recall that R is a fixed commutative ring.) By [BH, §3.1], the family $(x_C)_{C \in \mathrm{Comp}(n)}$ is free over R. Let

$$R\Sigma'(W_n) = \mathop{\oplus}_{C \in \mathrm{Comp}(n)} Rx_C \qquad \subset RW_n.$$

For simplification, we set $\Sigma'(W_n) = \mathbb{Z}\Sigma'(W_n)$, so that $R\Sigma'(W_n) = R \otimes_{\mathbb{Z}} \Sigma'(W_n)$.

Remark - The algebra $R\Sigma'(W_n)$ is nothing else but the algebra constructed by Mantaci and Reutenauer [MR] by combinatorial methods (see [BH, Remark of Subsection 3.1] for the identification). $\square$

Let $(\xi_C)_{C \in \mathrm{Comp}(n)}$ denote the basis $\mathrm{Hom}_R(R\Sigma'(W_n), R)$ dual to $(x_C)_{C \in \mathrm{Comp}(n)}$. In other words, we have, for every $x \in R\Sigma'(W_n)$,

$$x = \sum_{C \in \mathrm{Comp}(n)} \xi_C(x) x_C.$$

We now define

$$\theta_n^R : R\Sigma'(W_n) \longrightarrow R \operatorname{Irr} W_n$$

as the unique R-linear map such that

$$\theta_n^R(x_C) = \mathrm{Ind}_{W_C}^{W_n} 1_C$$

for every $C \in \mathrm{Comp}(n)$. Here, 1_C is the trivial character of W_C. We denote by ε_C the sign character of W_C. We can now recall the following result.

Theorem B [BH, Theorem 3.7].

(a) *$R\Sigma'(W_n)$ is a unitary sub-R-algebra of RW_n.*

(b) *$\theta_n^R : R\Sigma'(W_n) \to R \operatorname{Irr} W_n$ is a morphism of R-algebras.*

(c) *θ_n^R is surjective and* $\operatorname{Ker} \theta_n^R = \displaystyle\sum_{C \equiv D} R(x_C - x_D)$.

(d) *If K is a field of characteristic 0, then $\operatorname{Ker} \theta_n^K$ is the radical of the K-algebra $K\Sigma'(W_n)$.*

Let $\mathrm{Comp}^+(n)$ be the set of compositions of n. A signed composition $C = (c_1, \dots, c_r)$ is called *semi-positive* (resp. *parabolic*) if $c_i \geqslant -1$ (resp. $c_i < 0$) for every $i \geqslant 1$ (resp. for every $i \geqslant 2$). Note that C is parabolic if and only if W_C is a standard parabolic subgroup of W (i.e. if and only if $S_C \subset S_n$). We denote by $\mathrm{Comp}_{\mathrm{par}}(n)$ the set of parabolic compositions of n. Let

$$R\Sigma(W_n) = \mathop{\oplus}_{C \in \mathrm{Comp}_{\mathrm{par}}(n)} Rx_C$$

and

$$R\Sigma(\mathfrak{S}_n) = \mathop{\oplus}_{C \in \mathrm{Comp}^+(n)} Rx_C.$$

Then $R\Sigma(W_n)$ and $R\Sigma(\mathfrak{S}_n)$ are sub-R-algebras of $R\Sigma'(W_n)$: $R\Sigma(W_n)$ is the Solomon descent algebra of W_n (see [S] for the definition of Solomon descent algebras of finite Coxeter groups) and it is easy to check [BH, §3.2] that $R\Sigma(\mathfrak{S}_n)$ is the Solomon descent algebra of $\mathfrak{S}_n = W_{-n}$.

The restriction of θ_n^R to $R\Sigma(W_n)$ is equal to the classical Solomon homomorphism. On the other hand, the canonical surjective morphism $W_n \to \mathfrak{S}_n$ induces an injective morphism of algebras $R \operatorname{Irr} \mathfrak{S}_n \hookrightarrow R \operatorname{Irr} W_n$. We view $R \operatorname{Irr} \mathfrak{S}_n$ naturally as a subalgebra of $R \operatorname{Irr} W_n$ through this morphism. Then the image, through θ_n^R, of an element of $R\Sigma(\mathfrak{S}_n)$ belongs to $R \operatorname{Irr} \mathfrak{S}_n$. Also, the restriction of θ_n^R to a morphism (still denoted by θ_n^R) of algebras $R\Sigma(\mathfrak{S}_n) \to R \operatorname{Irr} \mathfrak{S}_n$ is again equal to the classical Solomon homomorphism. By construction, the diagram

(1.3)
$$\begin{tikzcd} R\Sigma(\mathfrak{S}_n) \arrow[r, hook] \arrow[d, "\theta_n^R"'] & R\Sigma'(W_n) \arrow[d, "\theta_n^R"] \\ R \operatorname{Irr} \mathfrak{S}_n \arrow[r, hook] & R \operatorname{Irr} W_n \end{tikzcd}$$

is commutative [BH, Diagram 3.4].

1.G. On the multiplication in $R\Sigma'(W_n)$. By Theorem B, $R\Sigma'(W_n)$ is a sub-R-algebra of RW_n and θ_n^R is a morphism of algebras. However, the multiplication in $R\Sigma'(W_n)$ is not described. In fact, it turns out that its description is much more complicated than the multiplication in the Solomon descent algebra. Theoretically, it is possible to extract from the proof of [BH, Theorem 3.7] an inductive process for this multiplication. We shall not do it here. We shall just give some easy consequences of this inductive process.

First, if $\mathcal{F}$ is a subset of $\operatorname{Comp}(n)$, we set

$$R\Sigma'_{\mathcal{F}}(W_n) = \bigoplus_{C \in \mathcal{F}} Rx_C.$$

For instance, $R\Sigma(W_n) = R\Sigma'_{\operatorname{Comp}_{\mathrm{par}}(n)}(W_n)$ and $R\Sigma(\mathfrak{S}_n) = R\Sigma'_{\operatorname{Comp}^+(n)}(W_n)$.

We shall now describe an order $\preccurlyeq$ on $\operatorname{Comp}(n)$ which is finer than $\subset$. Let C and D be two signed composition of n. We write $C \preccurlyeq D$ if one of the following two conditions is satisfied:

(1) $C \subset D$.
(2) $C \subset D^+$ and $\lg(C) > \lg(D)$ and $\lg^-(C) \geqslant \lg^-(D)$.

One can easily check that it defines an order $\preccurlyeq$ on $\operatorname{Comp}(n)$ (see Remark 1.2). We set

$$\mathcal{F}_{\prec D} = \{C \in \operatorname{Comp}(n) \mid C \prec D\}$$

and
$$\mathcal{F}_{\preccurlyeq D} = \{C \in \operatorname{Comp}(n) \mid C \preccurlyeq D\}.$$

For simplification, we set

$$R\Sigma'_{\prec D}(W_n) = R\Sigma'_{\mathcal{F}_{\prec D}}(W_n) \text{ and } R\Sigma'_{\preccurlyeq D}(W_n) = R\Sigma'_{\mathcal{F}_{\preccurlyeq D}}(W_n).$$

REMARK 1.4 - If $C \preccurlyeq D$, then $C^+ \subset D^+$, $\lg(C) \geqslant \lg(D)$ and $\lg^-(C) \geqslant \lg^-(D)$. □

We shall now describe a preorder $\subset_{\boldsymbol{\lambda}}$ on $\mathrm{Comp}(n)$. First, note that the order $\subset$ on $\mathrm{Comp}(n)$ induces an order on $\mathrm{Bip}(n)$ which we still denote by $\subset$. If C and D are two signed compositions of n, we then write $C \subset_{\boldsymbol{\lambda}} D$ if $\boldsymbol{\lambda}(C) \subset \boldsymbol{\lambda}(D)$. In other words, $C \subset_{\boldsymbol{\lambda}} D$ if and only if W_C is contained in some conjugate of W_D. We write $C \subsetneq_{\boldsymbol{\lambda}} D$ if $\boldsymbol{\lambda}(C) \subsetneq \boldsymbol{\lambda}(D)$. Similarly as above, we set

$$\mathcal{F}_{\subsetneq_{\boldsymbol{\lambda}} D} = \{C \in \mathrm{Comp}(n) \mid C \subsetneq_{\boldsymbol{\lambda}} D\}$$

and

$$\mathcal{F}_{\subset_{\boldsymbol{\lambda}} D} = \{C \in \mathrm{Comp}(n) \mid C \subset_{\boldsymbol{\lambda}} D\}.$$

For simplification, we set

$$R\Sigma'_{\subsetneq_{\boldsymbol{\lambda}} D}(W_n) = R\Sigma'_{\mathcal{F}_{\subsetneq_{\boldsymbol{\lambda}} D}}(W_n) \qquad \text{and} \qquad R\Sigma'_{\subset_{\boldsymbol{\lambda}} D}(W_n) = R\Sigma'_{\mathcal{F}_{\subset_{\boldsymbol{\lambda}} D}}(W_n)$$

REMARK 1.5 - It is easily checked that $\subset_{\boldsymbol{\lambda}}$ is a preorder on $\mathrm{Comp}(n)$ and that the equivalence relation associated to the preorder $\subset_{\boldsymbol{\lambda}}$ is exactly the relation $\equiv$. □

We now recall some notation from [BH, Proposition 2.13]. If C and D are two signed compositions of n, we set

$$X_{CD} = X_C^{-1} \cap X_D.$$

Moreover, if $d \in X_{CD}$, we denote by $C \cap {}^d D$ the unique signed composition of n such that $W_C \cap {}^d W_D = W_{C \cap {}^d D}$. If C, $C' \subset D$, we set $X_C^D = X_C \cap W_D$, $x_C^D = \sum_{w \in X_C^D} w \in RW_D$, $X_{CC'}^D = X_{CC'} \cap W_D$, $R\Sigma'(W_D) = \oplus_{C \subset D} Rx_C^D$ and we define $\theta_D^R : R\Sigma'(W_D) \to R \operatorname{Irr} W_D$, $x_C^D \mapsto \mathrm{Ind}_{W_C}^{W_D} 1_C$. Then $R\Sigma'(W_D)$ is a sub-R-algebra of RW_D and θ_D^R is a surjective morphism of algebras. Moreover, if $D = (d_1, \dots, d_r)$, then

$$R\Sigma'(W_D) \simeq R\Sigma'(W_{d_1}) \otimes_R \cdots \otimes_R R\Sigma'(W_{d_r}),$$

where $\Sigma'(W_d) = \Sigma(\mathfrak{S}_{-d})$ if $d < 0$.

Proposition C (see [BH, Proof of Theorem 3.7]). *Let C and D be two signed compositions of n. Then*

(a) *There is a map $f_{CD} : X_{CD} \to \mathrm{Comp}(n)$ such that:*
  (1) *For every $d \in X_{CD}$, $f_{CD}(d) \subset D$ and $f_{CD}(d) \equiv_D {}^{d^{-1}}C \cap D$.*
  (2) $x_C x_D - \displaystyle\sum_{d \in X_{CD}} x_{f_{CD}(d)} \in R\Sigma'_{\subsetneq_{\boldsymbol{\lambda}} C}(W_n) \cap R\Sigma'_{\prec D}(W_n) \cap \operatorname{Ker} \theta_n^R$.

(b) *If C is parabolic or if D is semi-positive, then $f_{CD}(d) = {}^{d^{-1}}C \cap D$ for every $d \in X_{CD}$ and $x_C x_D = \displaystyle\sum_{d \in X_{CD}} x_{{}^{d^{-1}}C \cap D}$.*

Proof. In fact, (b) is proved in [BH, Example 3.2]. Let us now prove (a). We first need an easy lemma about double cosets representatives:

Lemma 1.6. *Let C, D and D' be three signed compositions of n such that $D \subset D'$. Let $\mathcal{E} = \{(d,e) \mid d \in X_{CD'}$ and $e \in X_{({}^{d^{-1}}C \cap D'),D}^{D'}\}$. Let $f : \mathcal{E} \to X_{CD}$ be the map defined by $de \in W_C f(d,e) W_D$. Then f is bijective and $W_{f(d,e)^{-1}C \cap D}$ is conjugate, inside W_D, to ${}^{(de)^{-1}}W_C \cap W_D$.*

Now, by arguing by induction on n as in [BH, Proof of Theorem 3.7] and by using Lemma 1.6, we are reduced to the case where $C = (k,l)$ with k, $l \geqslant 1$ and $k+l=n$ and $D = (-n)$. Then this follows from [BH, Example 2.25]. $\square$

1.H. Some morphisms of algebras $R\Sigma'(W_n) \to R$. If $\lambda \in \mathrm{Bip}(n)$, let $\pi_\lambda : \Sigma'(W_n) \to \mathbb{Z}$, $x \mapsto \theta_n(x)(\mathrm{cox}_\lambda)$. Recall that $\theta_n(x)$ is a $\mathbb{Z}$-linear combination of permutation characters, so $\theta_n(x)(w)$ lies in $\mathbb{Z}$. Moreover, π_λ does not depend on the choice of cox_λ in $\mathcal{C}(\lambda)$, and is a morphism of $\mathbb{Z}$-algebras. We denote by $\pi_\lambda^R : R\Sigma'(W_n) \to R$ the morphism of algebras $\mathrm{Id}_R \otimes_{\mathbb{Z}} \pi_\lambda$. We denote by $\mathcal{D}_\lambda^R$ the left $R\Sigma'(W_n)$-module whose underlying R-module is free of rank one and on which $R\Sigma'(W_n)$ acts through π_λ^R. If K is a field, then $\mathcal{D}_\lambda^K$ is a simple $K\Sigma'(W_n)$-module.

If C and D are two signed compositions of n, let $X_{CD}^{\subset} = \{d \in X_{CD} \mid {}^{d^{-1}}W_C \subset W_D\}$. Then

$$\pi_{\boldsymbol{\lambda}(C)}(x_D) = |X_{CD}^{\subset}|. \tag{1.7}$$

Proof. By definition, we have

$$\pi_{\boldsymbol{\lambda}(C)}(x_D) = \Big(\mathrm{Ind}_{W_D}^{W_n} 1_D\Big)(\mathrm{cox}_C) = \Big(\mathrm{Res}_{W_C}^{W_n} \mathrm{Ind}_{W_D}^{W_n} 1_D\Big)(\mathrm{cox}_C).$$

Therefore, by the Mackey formula,

$$\pi_{\boldsymbol{\lambda}(C)}(x_D) = \sum_{d \in X_{CD}} \Big(\mathrm{Ind}_{W_{C \cap {}^dD}}^{W_C} 1_{C \cap {}^dD}\Big)(\mathrm{cox}_C).$$

But, by the argument in the proof of [BH, proposition 3.12], we get that cox_C lies in a subgroup of W_C conjugate to $W_{C \cap {}^dD}$ if and only if $C \cap {}^dD = C$. This shows the result. $\square$

1.I. Action of the normalizer. If C and D are two signed compositions of n, we set $X_{CD}^{\equiv} = \{d \in X_{CD} \mid W_C = {}^dW_D\}$. Then

$$X_{CD}^{\equiv} = \{d \in W_n \mid \Delta_C = d(\Delta_D)\}. \tag{1.8}$$

Proof. Let $\mathcal{X} = \{d \in W_n \mid \Delta_C = d(\Delta_D)\}$. Then it is clear that $\mathcal{X} \subset X_{CD}^{\equiv}$. Conversely, if $d \in X_{CD}^{\equiv}$, then $d(\Phi_D) = \Phi_C$. So $d(\Delta_D)$ is a basis of Φ_C, hence there exists $w \in W_C$ such that $d(\Delta_D) = w(\Delta_C)$. So $d^{-1}w \in X_C$ and $d^{-1} \in X_C$. So $w = 1$, as desired. $\square$

REMARK 1.9 - If D and D' are two signed compositions of n and if $d \in X_{DD'}$ is such that ${}^d W_{D'} = W_D$, then $d(\Delta_{D'}) = \Delta_D$ by 1.8 and

$$X_{D'} = X_D d \qquad \text{and} \qquad x_{D'} = x_D d.$$

Moreover, for every $C \subset D'$, we have $dx_C^{D'} d^{-1} = x_{{}^dC}^{D}$ (here, note that $d \in X_{DC}$, and we denote for simplification ${}^dC \cap D$ by ${}^dC$ because ${}^dW_C \cap W_D = {}^dW_C$). So conjugacy by d induces a morphism of algebras $d_* : R\Sigma'(W_{D'}) \to R\Sigma'(W_D)$. $\square$

If $C \equiv D$, then $X_{CD}^{\equiv} = X_{CD}^{\subset}$. If $D \in \mathrm{Comp}(n)$, we define $\mathcal{W}(D) = X_{DD}^{\subset}$.

Lemma 1.10. *Let D be a signed compositions of n. Then:*

(a) $\mathcal{W}(D) = \{w \in W_n \mid w(\Delta_D) = \Delta_D\}$.
(b) *$\mathcal{W}(D)$ is a subgroup of $N_{W_n}(W_D)$.*
(c) *The natural map $\mathcal{W}(D) \to N_{W_n}(W_D)/W_D$ is an isomorphism of groups.*
(d) $N_{W_n}(W_D) = \mathcal{W}(D) \ltimes W_D$.
(e) *If $C \in \mathrm{Comp}(n)$, then $|\mathcal{W}(D)|$ divides $|X_{CD}^{\subset}|$.*

Proof. (a), (b), (c) and (d) follow immediately from 1.8. Let us now prove (e). First, by 1.7, $|X_{CD}^{\subset}|$ is equal to the number of fixed points of cox_C in its action on W_n/W_D by left multiplication. But $\mathcal{W}(D)$ acts on W_n/W_D by right translation and this action commutes with the left action of cox_C. Therefore, $\mathcal{W}(D)$ permutes the fixed points of cox_C. Since $\mathcal{W}(D)$ acts freely on W_n/W_D, (e) follows. $\square$

2. ON THE IDEALS OF $R\Sigma'(W_n)$

This section is inspired by [BP, §3.A]. We shall define some families of left, right and two-sided ideals of $R\Sigma'(W_n)$ related to the order $\preccurlyeq$ and the preorder $\subset_{\boldsymbol{\lambda}}$ defined in the previous section. We need the following definition: if $x \in R\Sigma'(W_n)$, the *support of x* (denoted by $\mathrm{Supp}(x)$) is the subset of $\mathrm{Comp}(n)$ defined by

$$\mathrm{Supp}(x) = \{C \in \mathrm{Comp}(n) \mid \xi_C(x) \neq 0\}.$$

2.A. Some left ideals. A subset $\mathcal{F}$ of $\mathrm{Comp}(n)$ is called *left-saturated* if, for every $D \in \mathcal{F}$ and every $C \in \mathrm{Comp}(n)$ such that $C \preccurlyeq D$, we have $C \in \mathcal{F}$. By Proposition C (a), if $\mathcal{F}$ is left-saturated, then $R\Sigma'_{\mathcal{F}}(W_n)$ is a left ideal of $R\Sigma'(W_n)$.

If $x \in R\Sigma'(W_n)$, we set

$$\mathrm{Sat}_l(x) = \{C \in \mathrm{Comp}(n) \mid \exists\, D \in \mathrm{Supp}(x),\ C \preccurlyeq D\}.$$

Then $\mathrm{Sat}_l(x)$ is the minimal left-saturated subset of $\mathrm{Comp}(n)$ containing the support of x. By the previous remark,

(2.1) $$R\Sigma'(W_n)x \subset R\Sigma'_{\mathrm{Sat}_l(x)}(W_n).$$

EXAMPLE 2.2 - If $D \in \mathrm{Comp}(n)$, then $\mathcal{F}_{\preccurlyeq D}$ and $\mathcal{F}_{\prec D}$ are left-saturated. In fact, $\mathcal{F}_{\preccurlyeq D} = \mathrm{Sat}_l(x_D)$. Consequently, $R\Sigma'_{\preccurlyeq D}(W_n)$ and $R\Sigma'_{\prec D}(W_n)$ are left ideals of $R\Sigma'(W_n)$. Note that $R\Sigma'_{\preccurlyeq D}(W_n)/R\Sigma'_{\prec D}(W_n)$ is a left $R\Sigma'(W_n)$-module which is free of rank 1 over R (it is generated by the image of x_D). The action of $R\Sigma'(W_n)$ on this module is described in the next proposition. □

Proposition 2.3. *Let D be a signed composition of n and let $x \in \Sigma'(W_n)$. Then*

$$xx_D - \pi^R_{\boldsymbol{\lambda}(D)}(x)x_D \in R\Sigma'_{\prec D}(W_n).$$

In other words, $R\Sigma'_{\preccurlyeq D}(W_n)/R\Sigma'_{\prec D}(W_n) \simeq \mathcal{D}^R_{\boldsymbol{\lambda}(D)}$.

Proof. By Proposition C, we only need to show that $\xi_D(xx_D) = \pi_{\boldsymbol{\lambda}(D)}(x)$ for every $x \in R\Sigma'(W_n)$. Let $C \in \mathrm{Comp}(n)$. By Proposition C, we have

$$\xi_D(x_C x_D) = |\{d \in X_{CD} \mid W_D \subset {}^{d^{-1}}W_C\}| = |X^{\subset}_{CD}|.$$

So the result follows from 1.7. □

The next result follows immediately from Proposition 2.3.

Corollary 2.4. *Let $\mathcal{F}$ be a left-saturated subset of* $\mathrm{Comp}(n)$ *and let $\chi_{\mathcal{F}}$ denote the character of the left $K\Sigma'(W_n)$-module $K\Sigma'_{\mathcal{F}}(W_n)$. then*

$$\chi_{\mathcal{F}} = \sum_{C \in \mathcal{F}} \pi^K_{\boldsymbol{\lambda}(C)}.$$

If $a \in R\Sigma'(W_n)$, we denote by $f_a(T) \in R[T]$ its minimal polynomial. Let $\gamma_a : R\Sigma'(W_n) \to R\Sigma'(W_n)$, $x \mapsto ax$ be the left multiplication by a and let Γ_a be the matrix of γ_a in the basis $(x_C)_{C\in\mathrm{Comp}(n)}$. Then f_a is the minimal polynomial of γ_a (or of the matrix Γ_a). By 2.3, Γ_a is triangular (with respect to the order $\preccurlyeq$ on $\mathrm{Comp}(n)$) and its characteristic polynomial is

$$\prod_{C \in \mathrm{Comp}(n)} (T - \pi^R_{\boldsymbol{\lambda}(C)}(a)). \tag{2.5}$$

In particular:

Corollary 2.6. *f_a is split over R.*

2.B. Some right ideals. A subset $\mathcal{F}$ of $\mathrm{Comp}(n)$ is called *right-saturated* if, for every $D \in \mathcal{F}$ and every $C \in \mathrm{Comp}(n)$ such that $C \subset_{\boldsymbol{\lambda}} D$, we have $C \in \mathcal{F}$. If $\mathcal{F}$ is right-saturated, then $R\Sigma'_{\mathcal{F}}(W)$ is a right ideal of $R\Sigma'(W_n)$ (see Proposition D (a)). It must be noticed that, by opposition with the case of the classical Solomon algebra [BP, §3.B], $R\Sigma'_{\mathcal{F}}(W)$ is not necessarily a two-sided ideal of $R\Sigma'(W_n)$ (see Example 2.8 below).

EXAMPLE 2.7 - If $D \in \mathrm{Comp}(n)$, then $\mathcal{F}_{\subset_{\boldsymbol{\lambda}} D}$ and $\mathcal{F}_{\subsetneq_{\boldsymbol{\lambda}} D}$ are left-saturated. In fact, $\mathcal{F}_{\subset_{\boldsymbol{\lambda}} D} = \mathrm{Sat}_r(x_D)$. Consequently, $R\Sigma'_{\subset_{\boldsymbol{\lambda}} D}(W_n)$ and $R\Sigma'_{\subsetneq_{\boldsymbol{\lambda}} D}(W_n)$ are right ideals of $R\Sigma'(W_n)$. □

Example 2.8 - Assume here that $n=2$. Then $R\Sigma_{\subset_{\boldsymbol{\lambda}}(-2)}(W_2)=Rx_{(-2)}\oplus Rx_{(-1,-1)}$ is not a two-sided ideal of $\Sigma'_R(W_n)$ because

$$x_{(1,1)}x_{(-2)}=x_{(-1,-1)}+x_{(-1,1)}-x_{(1,-1)}.\ \square$$

If $x\in R\Sigma'(W_n)$, we set

$$\mathrm{Sat}_r(x)=\{C\in\mathrm{Comp}(n)\mid \exists\ D\in\mathrm{Supp}(a),\ C\subset_{\boldsymbol{\lambda}} D\}.$$

Then $\mathrm{Sat}_r(x)$ is the minimal right-saturated subset of $\mathrm{Comp}(n)$ containing the support of x. By the previous remark,

$$\textbf{(2.9)}\qquad xR\Sigma'(W_n)\subset R\Sigma'_{\mathrm{Sat}_r(x)}(W_n).$$

We shall construct in Section 3 a class of elements x for which equality holds in 2.9.

2.C. Some two-sided ideals. A subset $\mathcal{F}$ of $\mathrm{Comp}(n)$ is called *saturated* if it is left-saturated and right-saturated. If $\mathcal{F}$ is saturated, then $R\Sigma'_{\mathcal{F}}(W_n)$ is a two-sided ideal of $R\Sigma'(W_n)$.

Example 2.10 - If $k\geqslant 0$, we set $\mathcal{F}_k(n)=\{C\in\mathrm{Comp}(n)\mid \lg(C)\geqslant k+1\}$ and $\mathcal{F}_k^-(n)=\{C\in\mathrm{Comp}(n)\mid \lg(C)+\lg^-(C)\geqslant k+1\}$. Then, by the Remarks 1.2 and 1.4, $\mathcal{F}_k(n)$ and $\mathcal{F}_k^-(n)$ are saturated subsets of $\mathrm{Comp}(n)$. $\square$

3. Positivity properties

In this section, and only in this section, we assume that K is an ordered (for instance $K=\mathbb{Q}$ or $K=\mathbb{R}$). Recall that this implies that K has characteristic 0. We shall now construct a class of elements of $K\Sigma'(W_n)$ for which equality holds in 2.9. We denote by $K\Sigma'(W_n)^+$ the set of elements $a\in K\Sigma'(W_n)$ such that $\xi_C(a)\geqslant 0$ for every $C\in\mathrm{Comp}(n)$. Note that $x_C\in K\Sigma'(W_n)^+$ for any $C\in\mathrm{Comp}(n)$. If a and b are two elements of $K\Sigma'(W_n)^+$, then

$$\textbf{(3.1)}\qquad a+b\in K\Sigma'(W_n)^+.$$

However, contrarily to the case of Solomon algebras [BP, 3.2], it might happen that $ab\notin K\Sigma'(W_n)^+$ (see example 2.8). However, the analogue of [BP, First statement of Proposition 3.6] holds:

Proposition 3.2. *Assume that K is an ordered field. Let $a\in K\Sigma'(W_n)^+$. Then*

$$aK\Sigma'(W_n)=K\Sigma'_{\mathrm{Sat}_r(a)}(W_n).$$

Proof. Let $\mathcal{F} = \mathrm{Sat}_r(a)$. By 2.9, we have $aK\Sigma'(W_n) \subset K\Sigma'_{\mathcal{F}}(W_n)$. We shall show by induction on $C \in \mathcal{F}$ (induction with respect to the order $\preccurlyeq$) that $x_C \in aK\Sigma'(W_n)$. For this, we may, and we will, assume that $a \neq 0$.

First, if $C = (-1, -1, \dots, -1)$, then $x_C = \sum_{w \in W_n} w$ so

$$ax_C = \Big(\sum_{D \in \mathrm{Comp}(n)} \xi_D(a)|X_D| \Big) x_C.$$

Since $a \neq 0$ and $a \in K\Sigma'(W_n)^+$, we have by definition $\sum_{D \in \mathrm{Comp}(n)} \xi_D(a)|X_D| > 0$. So $x_{(-1,-1,\dots,-1)} \in aK\Sigma'(W_n)$.

Now, let $C \in \mathcal{F}$ and assume that, if $C' \in \mathcal{F}$ is such that $C' \prec C$, then $x_{C'} \in aK\Sigma'(W_n)$. Then, by Propositions C and 2.3, we have

$$ax_C - \pi_{\boldsymbol{\lambda}(C)}(a)x_C \in K\Sigma'_{\prec C}(W_n).$$

But, by the induction hypothesis, we have that $K\Sigma'_{\subset_{\boldsymbol{\lambda}} C}(W_n) \subset aK\Sigma'(W_n)$. So it remains to show that $\pi_{\boldsymbol{\lambda}(C)}(a) \neq 0$. But,

$$\pi_{\boldsymbol{\lambda}(C)}(a) = \sum_{D \in \mathrm{Supp}(a)} \xi_D(a)\pi_{\boldsymbol{\lambda}(C)}(x_D).$$

Since $\xi_D(a) > 0$ and $\pi_{\boldsymbol{\lambda}(C)}(x_D) \geqslant 0$ for every $D \in \mathrm{Supp}(a)$, it remains to show that there exists $D \in \mathrm{Supp}(a)$ such that $\pi_{\boldsymbol{\lambda}(C)}(x_D) > 0$. But, by the definition of $\mathcal{F}$, there exists $D \in \mathrm{Supp}(a)$ such that W_C is contained in a conjugate of W_D. So, for this particular D, we have that cox_C is contained in a conjugate of W_D. So $\pi_{\boldsymbol{\lambda}(C)}(x_D) = \mathrm{Ind}_{W_D}^{W_n}(\mathrm{cox}_C) \geqslant 1$ and the proof of the proposition is complete. $\square$

The next four corollaries must be compared with [BP, Corollaries 4.7, 3.8 and 3.12 and Proposition 3.10].

Corollary 3.3. *Assume that K is an ordered field. Let $a \in K\Sigma'(W_n)^+$. Then a is invertible in $K\Sigma'(W_n)$ if and only if $\xi_n(a) > 0$.*

Corollary 3.4. *Assume that K is an ordered field. Let $a_1, \dots, a_r \in K\Sigma'(W_n)^+$. Then $a_1 + \cdots + a_r \in K\Sigma'(W_n)^+$ and*

$$a_1K\Sigma'(W_n) + \cdots + a_rK\Sigma'(W_n) = (a_1 + \cdots + a_r)K\Sigma'(W_n).$$

The proof of the next corollary follows an argument of Atkinson [A].

Corollary 3.5. *Assume that K is an ordered field. Let $a \in K\Sigma'(W_n)^+$ and let r be a non-zero natural number. Then:*

(a) $\mathrm{val}\, f_a \leqslant 1$.
(b) $a^r K\Sigma'(W_n) = aK\Sigma'(W_n)$.
(c) $K\Sigma'(W_n)a^r = K\Sigma'(W_n)a$.

Proof. Recall that f_a is the minimal polynomial of a. We first prove (b). It is sufficient to show the result for $r = 2$. Let $m : aK\Sigma'(W_n) \to aK\Sigma'(W_n)$, $x \mapsto ax$. Then, by Proposition 3.2, we have $aK\Sigma'(W_n) = K\Sigma'_{\mathrm{Sat}_r(a)}(W_n)$. But, by the proof of Proposition 3.2, we have $\pi_{\boldsymbol{\lambda}(C)}(a) > 0$ for every $C \in \mathrm{Sat}_r(a)$. Therefore, by Proposition 2.3, the matrix of m in the basis $(x_C)_{C\in\mathrm{Sat}_r(a)}$ is triangular (with respect to the order $\preccurlyeq$) and has positive diagonal coefficients. So it is invertible. This shows (b).

(a) By (b), the minimal polynomial $f \in K[T]$ of m has a non-zero constant term. But, $f(a)a = f(m)(a) = 0$. Therefore, f_a divides $Tf(T)$. This shows (a).

(c) Now, by (a), we have that $a \in K[a]a^2$. So $a \in K\Sigma'(W_n)a^2$, as desired. ☐

Recall that γ_a denote the left multiplication $K\Sigma'(W_n) \to K\Sigma'(W_n)$, $x \mapsto ax$. Let $\delta_a : K\Sigma'(W_n) \to K\Sigma'(W_n)$, $x \mapsto xa$ denote the right multiplication by a.

Corollary 3.6. *Assume that K is an ordered field. Let $a \in K\Sigma'(W_n)^+$. Then:*

(a) $\mathrm{Ker}\,\gamma_a \oplus \mathrm{Im}\,\gamma_a = K\Sigma'(W_n)$.
(b) $\mathrm{Ker}\,\delta_a \oplus \mathrm{Im}\,\delta_a = K\Sigma'(W_n)$.

Proof. (a) For dimension reasons, it is sufficient to prove that $\mathrm{Ker}\,\gamma_a \cap \mathrm{Im}\,\gamma_a = 0$. Let $x \in \mathrm{Ker}\,\gamma_a \cap \mathrm{Im}\,\gamma_a$. Then $ax = 0$ and there exists $y \in K\Sigma'(W_n)$ such that $x = ay$. So $a^2y = 0$. Therefore, $a^ry = 0$ for every $r \geqslant 2$. But $a \in \sum_{r\geqslant 2} Ka^r$ by Corollary 3.5 (a). So $ay = 0$. In other words, $x = 0$, as desired. The proof of (b) is similar. ☐

REMARK 3.7 - By opposition with the case of Solomon descent algebras, it may happen that f_{x_C} is not square-free (compare with [BP, Proposition 3.10]). For instance, if $n = 4$ and if $K = \mathbb{Q}$, we have

$$f_{x_{(-3,1)}}(T) = T(T-2)(T-4)(T-8)^2(T-32).$$

This computation has been done using `CHEVIE` [Chevie]. □

We close this section by a result on the centralizers of positive elements (compare with [BP, Corollary 3.12]: the proof presented here is really different):

Proposition 3.8. *Assume that K is an ordered field. Let $a \in K\Sigma'(W_n)^+$ and r be a non-zero natural number. Then $Z_{K\Sigma'(W_n)}(a) = Z_{K\Sigma'(W_n)}(a^r)$.*

Proof. Let $A = \mathrm{End}_K\, K\Sigma'(W_n)$. Let $\gamma : K\Sigma'(W_n) \to A$, $x \mapsto \gamma_x$. It is an injective homomorphism of algebras. Therefore, $Z_{K\Sigma'(W_n)}(a) = \gamma^{-1}(Z_A(\gamma_a))$. So, in order to prove the proposition, we only need to prove that $Z_A(\gamma_a) = Z_A(\gamma_a^r)$.

Let $A' = \mathrm{End}_K(\mathrm{Ker}\,\gamma_a)$ and $A'' = \mathrm{End}_K(\mathrm{Im}\,\gamma_a)$. Then, by Corollary 3.6, $A' \oplus A''$ is a sub-K-algebra of A and $Z_A(\gamma_a)$ is contained in $A' \oplus A''$. Let γ'' denote the restriction of γ_a to $\mathrm{Im}\,\gamma_a$. Then $Z_A(\gamma_a) = A' \oplus Z_{A''}(\gamma'')$. Since $\mathrm{Ker}\,\gamma_a^r = \mathrm{Ker}\,\gamma_a$ and $\mathrm{Im}\,\gamma_a^r = \mathrm{Im}\,\gamma_a$, we only need to prove that $Z_{A''}(\gamma'') = Z_{A''}(\gamma''^r)$. But, by Proposition 3.2 and its proof, $(x_C)_{C\in\mathrm{Sat}_r(a)}$ is a basis of $\mathrm{Im}\,\gamma_a$ and the matrix of γ'' in this basis is triangular (with

respect to the order $\preccurlyeq$) with positive coefficients on the diagonal. So the proposition follows from Lemma 3.9 below. $\square$

Lemma 3.9. *Let m be a non-zero natural number. Let $M = (m_{ij}) \in \mathrm{Mat}_m(K)$ be an upper triangular $m \times m$ matrix such that $m_{ii} > 0$ for every $i \in \{1, 2, \ldots, m\}$. Then*

$$Z_{\mathrm{Mat}_m(K)}(M) = Z_{\mathrm{Mat}_m(K)}(M^r)$$

for every $r \geqslant 1$.

Proof of Lemma 3.9. Let $E = \mathrm{Mat}_m(K)$. Since M is invertible, we can write $M = SU = US$ where S (resp. U) is a diagonalizable (resp. unipotent) matrix. Then $Z_E(M) = Z_E(S) \cap Z_E(U)$. So it is sufficient to show that $Z_E(S) = Z_E(S^r)$ and $Z_E(U) = Z_E(U^r)$.

Since S is diagonalizable, we may assume that it is diagonal. Now the fact that $Z_E(S) = Z_E(S^r)$ follows from the fact that, if x, $y \in K$ are such that $x > 0$, $y > 0$ and $x^r = y^r$, then $x = y$ (because K is an ordered field).

Let us now show that $Z_E(U) = Z_E(U^r)$. Since K is an ordered field, its characteristic is zero. Let N be a nilpotent matrix such that $U = e^N$ (exponential). Then $Z_E(U) = Z_E(N)$ and, since $U^r = e^{rN}$, we have $Z_E(U^r) = Z_E(rN) = Z_E(N)$ (because $r \neq 0$ in K). $\square$

4. Action of the longest element

If $C \in \mathrm{Comp}(n)$, we denote by w_C the longest element of W_C. If $C \in \mathrm{Comp}^+(n)$, we denote by σ_C the longest element of $\mathfrak{S}_C$ (in other words, $\sigma_C = w_{-C}$). In particular, w_n is the longest element of W_n. Recall that $w_n \in R\Sigma'(W_n)$ (in fact $w_n \in R\Sigma(W_n)$) and that $\theta_n^R(w_n) = \varepsilon_n$, the sign character of W_n (see for instance [S]). Moreover, w_n is central in W_n, so it is central in $R\Sigma'(W_n)$.

First, note that

(4.1) $$\varepsilon_n(\mathrm{cox}\, C) = (-1)^{n - \mathrm{lg}^-(C)}.$$

Recall that the function $\mathrm{lg}^- : \mathrm{Comp}(n) \to \mathbb{N}$ has been defined in §1.C. In particular,

(4.2) $$\pi_\lambda(w_n) = (-1)^{n - \mathrm{lg}^-(\lambda)}$$

for all $\lambda \in \mathrm{Bip}(n)$.

From now on, and until the end of this section, we assume that 2 is invertible in R. Write

$$e_n^+ = \frac{1}{2}(1 + w_n) \qquad \text{and} \qquad e_n^- = \frac{1}{2}(1 - w_n).$$

Since w_n is central in W_n, e_n^+ and e_n^- are central idempotents of $K\Sigma'(W_n)$. Moreover, they are orthogonal and $e_n^+ + e_n^- = 1$.

We shall now describe a basis of $R\Sigma'(W_n)e_n^+$ and $R\Sigma'(W_n)e_n^-$. This is build on the same model as for the classical Solomon algebra [BP, §2.B]. First, note that

(4.3) $$w_n X_C = X_C w_C$$

for every signed composition C of n. This is proved as follows. An element $w \in W_n$ belongs to X_C if and only if $w(\alpha) > 0$ for every $\alpha \in \Phi_C^+$. In other words, $w \in w_n X_C$ if and only if $w(\alpha) < 0$ for every $\alpha \in \Phi_C^+$. A similar argument applies to show that $w \in X_C w_C$ if and only if $w(\alpha) < 0$ for every $\alpha \in \Phi_C^+$. This shows 4.3. Consequently,

(4.4) $$w_n x_C = x_C w_C.$$

Since 2 is invertible in R, we can define

$$x'_C = \sum_{\substack{D \in \mathrm{Comp}(n) \\ S_D \subset S_C}} \Bigl(-\frac{1}{2}\Bigr)^{|S_C|-|S_D|} x_D.$$

These elements have the following properties:

Proposition 4.5. *If 2 is invertible in R, then:*

(a) $(x'_C)_{C \in \mathrm{Comp}(n)}$ *is an R-basis of $R\Sigma'(W_n)$.*
(b) $\mathrm{Ker}\, \theta_n^R = \sum_{C \equiv D} R(x'_C - x'_D)$.
(c) $w_n x'_C = (-1)^{|S_C|} x'_C = (-1)^{n-\mathrm{lg}^-(C)} x'_C$.

Proof. (a) is trivial. Note for information that

$$x_C = \sum_{\substack{D \in \mathrm{Comp}(n) \\ S_D \subset S_C}} \Bigl(\frac{1}{2}\Bigr)^{|S_C|-|S_D|} x'_D.$$

(b) follows from Theorem B (c) and Remark 1.9.

Let us now prove (c). First, we set

$$\tilde{x}_C = \sum_{\substack{D \in \mathrm{Comp}(n) \\ S_D \subset S_C}} \Bigl(-\frac{1}{2}\Bigr)^{|S_C|-|S_D|} x_D^C.$$

Now, by 4.4, we get $w_n x_C = x_C w_C \tilde{x}_C$. But now, $\tilde{x}_C$ is an element of the classical Solomon descent algebra $R\Sigma(W_C)$ to which the result of [BP, 2.12] can be applied: we get $w_C \tilde{x}_C = (-1)^{|S_C|}\tilde{x}_C$. This shows the first equality. The second equality is easy from the definition of S_C. □

Corollary 4.6. *If 2 is invertible in R, then* $\dim_R R\Sigma'(W_n)e_n^+ = \dim_R R\Sigma'(W_n)e_n^- = 3^{n-1}$.

Proof. Let

$$\mathcal{C}^+ = \{C \in \mathrm{Comp}(n) \mid \mathrm{lg}^-(C) \equiv n \mod 2\}$$

and $$\mathcal{C}^- = \{C \in \mathrm{Comp}(n) \mid \mathrm{lg}^-(C) \equiv n+1 \mod 2\}.$$

Then, by Proposition 4.5 (a) and (c), we have $\dim_R R\Sigma'(W_n)e_n^? = |\mathcal{C}^?|$ for $? \in \{+,-\}$. It is now sufficient to show that $|\mathcal{C}^+| = |\mathcal{C}^-| = 3^{n-1}$. Since the number of compositions of n of length k is equal to $\binom{n-1}{k-1}$, we have

$$|\mathcal{C}^+| = \sum_{k=1}^{n} \binom{n-1}{k-1} 2^{k-1} = 3^{n-1},$$

and similarly for $|\mathcal{C}^-|$. ☐

EXAMPLE 4.7 - We consider in this example the elements of the form $x'_{\nu_1,\nu_2,\dots,\nu_n}$ of $R\Sigma'(W_n)$ where $\nu_i \in \{1,-1\}$. We shall show that they are quasi-idempotents. We first need some notation. Let $I_n^+ = \{1,2,\dots,n\}$. If $\sigma \in W_n$, let $\mathbf{I}(\sigma) = \{i \in I_n^+ \mid \sigma(i) > 0\}$. If $I \subset I_n^+$, we denote by $\mathbf{C}(I)$ the signed composition $(\nu_1,\dots,\nu_n)$, where $\nu_i = 1$ (respectively $\nu_i = -1$) if $i \in I$ (respectively $i \notin I$). We also define $\gamma_I : W_n \to R^\times$, $\sigma \mapsto (-1)^{|I|-|I\cap\mathbf{I}(\sigma)|}$. For instance, $\gamma_\varnothing = 1_n$ and $\gamma_{I_n^+}$ is a linear character of W_n (it will be denoted by γ_n for simplification). Note also that the restriction of γ_I to $\mathfrak{T}_n$ is always a linear character (the group $\mathfrak{T}_n$ has been defined in §1.B: it is the group generated by $t_1,\dots, t_n$). Moreover, if $\sigma \in \mathfrak{S}_n$ and $\tau \in W_n$, we have $\mathbf{I}(\sigma\tau) = \mathbf{I}(\tau)$, so $\gamma_I(\sigma\tau) = \gamma_I(\tau)$. Finally, we denote by $\mathfrak{S}_n(I)$ the stabilizer of I in $\mathfrak{S}_n$.

With this notation, we have

(4.8)
$$x'_{\mathbf{C}(I)} = \frac{1}{2^{|I|}} \sum_{\sigma\in W_n} \gamma_I(\sigma)\sigma.$$

Proof of 4.8. First, note that

$$x'_{\mathbf{C}(I)} = \sum_{J\subset I} \Bigl(-\frac{1}{2}\Bigr)^{|I|-|J|} x_{\mathbf{C}(J)}.$$

Now, let $\sigma \in W_n$ and $J \subset I_n^+$. Then $\sigma \in X_{\mathbf{C}(J)}$ if and only $J \subset \mathbf{I}(\sigma)$. Therefore, by the previous equality, the coefficient of σ in $x'_{\mathbf{C}(I)}$ is equal to

$$\sum_{J\subset I\cap\mathbf{I}(\sigma)} \Bigl(-\frac{1}{2}\Bigr)^{|I|-|J|} = \frac{(-1)^{|I|-|I\cap\mathbf{I}(\sigma)|}}{2^{|I|}},$$

as desired. ☐

If I and J are two subsets of I_n^+, then

(4.9)
$$x'_{\mathbf{C}(I)}x'_{\mathbf{C}(J)} = \begin{cases} 2^{n-|I|}|\mathfrak{S}_n(I)|x'_{\mathbf{C}(J)} & \text{if } |I| = |J|, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of 4.9. Let $e_I = \sum_{\tau\in\mathfrak{T}_n} \gamma_I(\tau)\tau$ and let $e = \sum_{\sigma\in\mathfrak{S}_n} \sigma$. Then, by 4.8, we have $x'_{\mathbf{C}(I)} = ee_I/2^{|I|}$. Moreover,

$$e_Ie_J = \begin{cases} 2^n e_I & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} x'_{\mathbf{C}(I)}x'_{\mathbf{C}(J)} &= \frac{1}{2^{|I|+|J|}}ee_I\Big(\sum_{\sigma\in\mathfrak{S}_n}\sigma\Big)e_J \\ &= \frac{1}{2^{|I|+|J|}}\sum_{\sigma\in\mathfrak{S}_n} e\sigma^{-1}e_I\sigma e_J \\ &= \frac{1}{2^{|I|+|J|}}\sum_{\sigma\in\mathfrak{S}_n} ee_{\sigma^{-1}(I)}e_J. \end{aligned}$$

If $|I| \neq |J|$, then $\sigma^{-1}(I) \neq J$ for every $\sigma \in \mathfrak{S}_n$. If $|I| = |J|$, then the number of elements $\sigma \in \mathfrak{S}_n$ such that $\sigma^{-1}(I) = J$ is equal to $|\mathfrak{S}_n(I)| = |I|!(n-|I|)!$. This shows the result in this last case. □

In particular, if $2|\mathfrak{S}_n(I)|$ is invertible in R, then $x'_{\mathbf{C}(I)}/(2^{n-|I|}|\mathfrak{S}_n(I)|)$ is an idempotent of $R\Sigma'(W_n)$. We shall show later that it is a primitive idempotent of $R\Sigma'(W_n)$ (see 8.4). A description of the module $\mathbb{Q}W_n x'_{\mathbf{C}(I)}$ will be given in Example 10.1.

Since $\gamma_{I_n^+} = \gamma_n$ is a linear character of W_n, we deduce immediately the following two properties of $x'_{1,1,\dots,1}$:

(4.10) $x'_{1,1,\dots,1}$ *is central in* $\mathbb{Q}W_n$*, hence is central in* $\mathbb{Q}\Sigma'(W_n)$*;*

(4.11) $$(x'_{1,1,\dots,1})^2 = n!\ x'_{1,1,\dots,1}.$$

In particular, if p does not divide $|W_n|$, then $x_{1,1,\dots,1}/n!$ is a primitive central idempotent of $\mathbb{Q}W_n$, hence is a primitive central idempotent of $\mathbb{Q}\Sigma'(W_n)$. □

5. Restriction morphisms between Mantaci-Reutenauer algebras

F. Bergeron, N. Bergeron, R.B. Howlett and D.E. Taylor [BBHT] have constructed so-called *restriction morphisms* between the Solomon algebra of a finite Coxeter group and the Solomon algebras of its standard parabolic subgroups. We shall construct here a restriction morphism $R\Sigma'(W_n) \to R\Sigma'(W_D)$ whenever D is a semi-positive signed compositions of n. It might be possible that such a morphism exists for every signed compositions, but we are not able to prove it (or to prove that there is no analogue in general). Most of the results of this section have an analogue in the context of Solomon's descent algebras [BP, §4].

5.A. Definition. We fix in this section a semi-positive signed composition D of n. We denote by $\mathrm{Res}_D : R\Sigma'(W_n) \to R\Sigma'(W_D)$ the unique R-linear map such that

$$\mathrm{Res}_D\, x_C = \sum_{d\in X_{CD}} x^D_{{}^{d^{-1}}C\cap D}$$

for every $C \in \mathrm{Comp}(n)$. If $C \subset D$ is a semi-positive signed composition, we define $\mathrm{Res}^D_C : R\Sigma'(W_D) \to R\Sigma'(W_C)$ similarly.

Proposition 5.1. *Let D be a semi-positive signed composition of n. Then:*

(a) *If $x \in \Sigma(W_n)$, then $x_D \operatorname{Res}_D(x) = x x_D$.*
(b) Res_D *is a morphism of algebras.*
(c) *If $C \subset D$ is also semi-positive, then $\operatorname{Res}_C^D \circ \operatorname{Res}_D = \operatorname{Res}_C$.*
(d) *The diagram*

$$\begin{tikzcd} R\Sigma'(W_n) \arrow[r, "\theta_n^R"] \arrow[d, "\operatorname{Res}_D"'] & R \operatorname{Irr} W_n \arrow[d, "\operatorname{Res}_{W_D}^{W_n}"] \\ R\Sigma'(W_D) \arrow[r, "\theta_D^R"] & R \operatorname{Irr} W_D \end{tikzcd}$$

is commutative.
(e) *If D' is another signed composition of n and if $d \in X_{DD'}$ is such that ${}^d S_{D'} = S_D$, then D' is semi-positive and $d_* \circ \operatorname{Res}_{D'} = \operatorname{Res}_D$.*

Proof. (a) follows from Proposition C (b). (b) and (c) follow from (a) and from the fact that the map $\mu_D : R\Sigma'(W_D) \to R\Sigma'(W_n)$, $x \mapsto x_D x$ is injective. (d) follows from the Mackey formula. (e) follows easily from Remark 1.9 and from (a). $\square$

By Remark 1.9, the group $\mathcal{W}(D)$ acts on $R\Sigma'(W_D)$. Moreover, by Proposition 5.1 (e), we have

(5.2) $$\operatorname{Im} \operatorname{Res}_D \subset R\Sigma'(W_D)^{\mathcal{W}(D)}.$$

Let $\operatorname{Comp}(D) = \{C \in \operatorname{Comp}(n) \mid C \subset D\}$. Write $D = (d_1, \dots, d_r)$ and $\operatorname{Bip}(D) = \operatorname{Bip}(d_1) \times \cdots \times \operatorname{Bip}(d_r)$: here, if $d < 0$, $\operatorname{Bip}(d)$ denotes the set of partitions of $-d$. If $C \in \operatorname{Comp}(D)$, we denote by $\boldsymbol{\lambda}_D(C)$ the element of $\operatorname{Bip}(D)$ defined in the natural way component by component. Then $\boldsymbol{\lambda}_D(C) = \boldsymbol{\lambda}_D(C')$ if and only if $C \equiv_D C'$. Therefore, the canonical injection $\operatorname{Comp}(D) \hookrightarrow \operatorname{Comp}(n)$ induces a unique map $\tau_D : \operatorname{Bip}(D) \to \operatorname{Bip}(n)$ such that $\tau_D(\boldsymbol{\lambda}_D(C)) = \boldsymbol{\lambda}(C)$.

Corollary 5.3. *Let D be a semi-positive signed composition of n and let $\lambda \in \Lambda_D$. Then $\pi_{\tau_D(\lambda)}^R = \pi_\lambda^R \circ \operatorname{Res}_D$.*

Proof. Let $(\xi_C^D)_{C \in \operatorname{Comp}(D)}$ denote the basis of $\operatorname{Hom}_R(R\Sigma'(W_D), R)$ dual to $(x_C^D)_{C \in \operatorname{Comp}(D)}$. Let C be a signed composition of n which is contained in D and let $\lambda = \boldsymbol{\lambda}_D(C) \in \operatorname{Bip}(D)$. Then, if $x \in R\Sigma'(W_D)$, we have by Proposition 2.3,

$$\pi_\lambda^R(x) = \xi_C^D(x x_C^D).$$

Therefore, if $x \in R\Sigma'(W_n)$, we have

$$\begin{aligned}\pi_\lambda^R(\mathrm{Res}_D\, x) &= \xi_C^D((\mathrm{Res}_D\, x)x_C^D)\\ &= \xi_C(x_D(\mathrm{Res}_D\, x)x_C^D)\\ &= \xi_C(xx_Dx_C^D)\\ &= \xi_C(xx_C)\\ &= \pi_{\boldsymbol{\lambda}(C)}(x).\end{aligned}$$

Now, the result follows from the fact that $\boldsymbol{\lambda}(C) = \tau_D(\lambda)$ by definition. □

We conclude this subsection by a result on the kernel of Res_D.

Proposition 5.4. *We have:*

(a) $\mathrm{Ker}(\mathrm{Res}_D) = \{x \in R\Sigma'(W_n) \mid xx_D = 0\}$.
(b) *If K is a field of characteristic zero, then* $K\Sigma'(W_D) = \mathrm{Ker}(\mathrm{Res}_D) \oplus K\Sigma'(W_n)x_D$.

Proof. (a) follows from Proposition 5.1 (a). To prove (b), we may, and we will, assume that $K = \mathbb{Q}$. Then, since $\mathbb{Q}$ is an ordered field, (b) follows from (a) and Corollary 3.6. □

5.B. Restriction to the Solomon algebra of $\mathfrak{S}_n$. If D is a semi-positive signed composition of n, then $\mathfrak{S}_{D^+}$ is a parabolic subgroup of $\mathfrak{S}_n$. So there is a restriction morphism $\mathrm{Res}_{D^+}^{\mathfrak{S}} : R\Sigma(\mathfrak{S}_n) \to R\Sigma(\mathfrak{S}_{D^+})$ which was constructed in [BBHT] (see also [BP, Proposition 4.1] for the proof of the fact that it is a morphism of algebras: it works as in the Proposition 5.1 above). Then the diagram

(5.5)
$$\begin{tikzcd} R\Sigma(\mathfrak{S}_n) \arrow[r, hook] \arrow[d, "\mathrm{Res}_{D^+}^{\mathfrak{S}}"'] & R\Sigma'(W_n) \arrow[d, "\mathrm{Res}_D"] \\ R\Sigma(\mathfrak{S}_{D^+}) \arrow[r, hook] & R\Sigma'(W_D) \end{tikzcd}$$

is commutative. Indeed, if $x \in R\Sigma(\mathfrak{S}_n)$, then, by definition, we have

$$x_D\,\mathrm{Res}_D(x) = xx_D = xx_{D^+}x_D^{D^+} = x_{D^+}\,\mathrm{Res}_{D^+}^{\mathfrak{S}}(x)x_D^{D^+}.$$

So it remains to show that, if $u \in R\Sigma(\mathfrak{S}_{D^+})$, then $ux_D^{D^+} = x_D^{D^+}u$. By direct product, we are reduce to prove this whenever $D = D^+$ (in which case it is trivial) or whenever $D = D^-$. In this last case, since D is semi-positive, we have $D^+ = (1)$ and the result follows from the fact that the algebra $R\Sigma'(W_1) = RW_1$ is commutative.

5.C. An example. Now, let us consider a particular semi-positive signed composition. If $D = (k, -1, -1, \ldots, -1)$, where $k \geqslant 1$, then Res_D induces in fact a morphism of algebras $\mathrm{Res}_k^n : R\Sigma'(W_n) \to R\Sigma'(W_k)$. Since D is also parabolic, we have that $\mathrm{Res}_k^n(R\Sigma(W_n)) \subset$

$R\Sigma(W_k)$, that the induced map $R\Sigma(W_n) \to R\Sigma(W_k)$ coincides with the map denoted by $\mathrm{Res}_{S_k}^{S_n}$ in [BP, §4.B] and that the diagram

(5.6)
$$\begin{tikzcd} R\Sigma(W_n) \arrow[r, hook] \arrow[d, "\mathrm{Res}_{S_k}^{S_n}"'] & R\Sigma'(W_n) \arrow[d, "\mathrm{Res}_k^n"] \\ R\Sigma(W_k) \arrow[r, hook] & R\Sigma'(W_k) \end{tikzcd}$$

is commutative. The next result can be compared with [BP, Proposition 4.15].

Proposition 5.7. *If $R = K$ is a field of characteristic zero, then Res_k^n is surjective.*

REMARK 5.8 - It is probable that the above proposition remains valid for every commutative ring R (i.e. for $R = \mathbb{Z}$). It has been checked for $n \leqslant 5$ using `CHEVIE` [Chevie]. □

Proof. By transitivity (see Proposition 5.1 (c)), we only need to prove that Res_{n-1}^n is surjective. We almost reproduce the argument in [BP, Proposition 4.15]. We have

$$X_{n-1,-1} = \{s_i s_{i+1} \dots s_{n-1} \mid 1 \leqslant i \leqslant n\} \coprod \{s_i s_{i-1} \dots s_1 t s_1 s_2 \dots s_{n-1} \mid 0 \leqslant i \leqslant n-1\}.$$

Therefore, if $d \in W_n$ and $i \in \{1, 2, \dots, n-1\}$ (resp. $i \in \{1, 2, \dots, n\}$) are such that $d^{-1} \in X_{n-1,-1}$, $\ell(ds_i) > \ell(d)$ (resp. $\ell(dt_i) > \ell(d)$), and $ds_i d^{-1} \in S'_{n-1}$ (resp. $dt_i d^{-1} \in S'_{n-1}$), then

$$(*) \qquad ds_i d^{-1} \in \{s_i, s_{i-1}\}$$

(resp.

$$(**) \qquad dt_i d^{-1} \in \{t_i, t_{i-1}\}.$$

We now define a total order $\trianglelefteq$ on $\mathrm{Comp}(n-1)$. Let C and D be two signed compositions of $n-1$. We write $C \trianglelefteq D$ if and only if one of the following two conditions are satisfied:

(1) $|S'_C| < |S'_D|$;
(2) $|S'_C| = |S'_D|$ and S'_C is smaller than S'_D for the lexicographic order induced by the order $t_1 < t_2 < \cdots < t_{n-1} < s_1 < \cdots < s_{n-2}$ on S'_{n-1}.

It follows immediately from $(*)$ and $(**)$ that

$$\mathrm{Res}_{n-1}^n x_{D \sqcup (-1)} \in \alpha_D x_D + \sum_{C \triangleleft D} K x_C$$

with $\alpha_D \in \mathbb{Z}$, $\alpha_D > 0$ (for every $D \in \mathrm{Comp}(n-1)$). Recall that the operation $\sqcup$ (concatenation) has been defined in §1.C. The proof of the proposition is complete. □

6. Simple modules, radical

Hypothesis and notation: *From now on, and until the end of this paper, we assume that $R = K$ is a field. We denote by p its characteristic ($p \geqslant 0$). We denote by $\mathrm{Bip}_{p'}(n)$ the set of bipartitions λ of n such that $o(\lambda)$ is invertible in K (recall that $o(\lambda)$ denotes the order of cox_λ). If $\lambda \in \mathrm{Bip}(n)$, we denote by $\lambda_{p'}$ the bipartition of n such that the p'-part of cox_λ is conjugate to $\mathrm{cox}_{\lambda_{p'}}$ (if $p = 0$, then $\lambda_{p'} = \lambda$).*

6.A. Simple modules. Since $\mathrm{Ker}\,\theta_n^K$ is a nilpotent two-sided ideal of $K\Sigma'(W_n)$, it is contained in the kernel of every simple representation of $K\Sigma'(W_n)$. Therefore, every simple representation factorizes (through θ_n^K) to a simple representation of $K\,\mathrm{Irr}\,W_n$. Since every irreducible character of W_n has value in $\mathbb{Z}$, and since θ_n^K is surjective by Theorem B (c), we get (see for instance [B2, Proposition 2.14 and Corollary 2.15]):

Proposition 6.1. *Let λ and μ be two bipartitions of n.*

(a) $\mathcal{D}_\lambda^K \simeq \mathcal{D}_\mu^K$ *if and only if* $\lambda_{p'} = \mu_{p'}$.
(b) $\{\mathcal{D}_\lambda^K \mid \lambda \in \mathrm{Bip}_{p'}(n)\}$ *is a set of representatives of isomorphy classes of simple left $K\Sigma'(W_n)$-modules.*

Corollary 6.2. *$K\Sigma'(W_n)$ is split.*

Corollary 6.3. $\mathrm{Irr}_K\, K\Sigma'(W_n) = \{\pi_\lambda^K \mid \lambda \in \mathrm{Bip}_{p'}(n)\}$.

The formula for the irreducible characters of $K\Sigma'(W_n)$ is given by 1.7.

6.B. Radical. The aim of this subsection is to describe the radical of $K\Sigma'(W_n)$ in full generality. If $p = 0$, then this is done in Theorem B (d). Let $\mathrm{Comp}_p(n) = \{C \in \mathrm{Comp}(n) \mid p \text{ divides } |\mathcal{W}(C)|\}$. The next result must be compared with [APVW, Theorem 3]:

Theorem 6.4. *If K is a field of characteristic p, then*

$$\mathrm{Rad}\, K\Sigma'(W_n) = \mathrm{Ker}\,\theta_n^K + \sum_{C \in \mathrm{Comp}_p(n)} K x_C.$$

Proof. Let $\mathcal{I} = \mathrm{Ker}\,\theta_n^K + \sum_{C \in \mathrm{Comp}_p(n)} K x_C$. By Proposition 6.1, we get that

$$\mathrm{Rad}\, K\Sigma'(W_n) = \bigcap_{\lambda \in \mathrm{Bip}(n)} \mathrm{Ker}\,\pi_\lambda^K.$$

Now, if $\lambda \in \mathrm{Bip}(n)$, then $\mathrm{Ker}\,\theta_n^K \subset \mathrm{Ker}\,\pi_\lambda^K$ and $x_C \subset \mathrm{Ker}\,\pi_\lambda^K$ for every $C \in \mathrm{Comp}_p(n)$ by 1.7 and by Lemma 1.10 (e). Therefore, $\mathcal{I} \subset \mathrm{Rad}\, K\Sigma'(W_n)$.

Now, let $x \in \mathrm{Rad}\, K\Sigma'(W_n)$. We want to prove that $x \in I$. Let $C \in \mathrm{Comp}(n)$ be maximal (for the preorder $\subset_{\boldsymbol{\lambda}}$) such that $\xi_C(x) \neq 0$. By an easy induction argument (on

the preorder $\subset_{\boldsymbol{\lambda}}$) we only need to prove that $x' = \sum_{C' \equiv C} \xi_{C'}(x) x_{C'}$ belongs to $\mathcal{I}$. Let $\lambda = \boldsymbol{\lambda}(C)$. Then, by Lemma 1.10 (d) and by 1.7, we have

$$0 = \pi_\lambda(x) = \pi_\lambda(x') = |\mathcal{W}(C)| \sum_{C' \equiv C} \xi_{C'}(x).$$

To cases may occur:

- If $C \in \mathrm{Comp}_p(n)$, then $x' \in \sum_{D \in \mathrm{Comp}_p(n)} K x_D \subset \mathcal{I}$.

- If $C \notin \mathrm{Comp}_p(n)$, then p does not divide $|\mathcal{W}(C)|$, so $\sum_{C' \equiv C} \xi_{C'}(x) = 0$. So $x' \in \operatorname{Ker} \theta_n^K \subset \mathcal{I}$. This completes the proof of the proposition. □

Corollary 6.5. $|\operatorname{Bip}(n)| = |\operatorname{Bip}_{p'}(n)| + |\boldsymbol{\lambda}(\mathrm{Comp}_p(n))|$.

Proof. By Proposition 6.4, we get that

$$\dim_K\bigl(\operatorname{Rad} K\Sigma'(W_n)\bigr) = \dim_K(\operatorname{Ker} \theta_n^K) + |\boldsymbol{\lambda}(\mathrm{Comp}_p(n))|.$$

On the other hand, we have

$$\dim_K K\Sigma'(W_n) = \dim_K(\operatorname{Ker} \theta_n^K) + |\operatorname{Bip}(n)|$$

and
$$\dim_K K\Sigma'(W_n) = \dim_K\bigl(\operatorname{Rad} K\Sigma'(W_n)\bigr) + |\operatorname{Bip}_{p'}(n)|.$$

(the last equality follows from Proposition 6.1 (b)). The corollary now follows from these observations. □

Note that the above corollary could have been proved directly by a pure combinatorial argument. Let us sketch it here. First, a bipartition $\lambda = (\lambda^+, \lambda^-)$ is said *p-regular* (respectively *p-singular*) if it does not belong (respectively if it belongs) to $\boldsymbol{\lambda}(\mathrm{Comp}_p(n))$. The set of p-regular partitions of n (which will be denoted by $\operatorname{Bip}_{p-\mathrm{reg}}(n)$) can be described more concretely as follows. If $i \geqslant 1$, we denote by $r_i^+(\lambda)$ (respectively $r_i^-(\lambda)$) the number of occurrences of i as a part of λ^+ (respectively λ^-). Similarly, if $C \in \mathrm{Comp}(n)$, we denote by $r_i^+(C)$ (respectively $r_i^-(C)$) the number of occurrences of i (respectively $-i$) as a part of C. In other words, $r_i^+(C) = r_i^+(\boldsymbol{\lambda}(C))$ and $r_i^-(C) = r_i^-(\boldsymbol{\lambda}(C))$. It is readily seen that

(6.6) $$\mathcal{W}(C) \simeq N_{W_n}(W_C)/W_C \simeq \mathfrak{S}_{r_1^+(C)} \times \cdots \times \mathfrak{S}_{r_n^+(C)} \times W_{r_1^-(C)} \times \cdots \times W_{r_n^-(C)}.$$

Consequently,

(6.7) $$\operatorname{Bip}_{2-\mathrm{reg}}(n) = \{\lambda \in \operatorname{Bip}(n) \mid \forall i \geqslant 1,\ r_i^+(\lambda) \leqslant 1 \text{ and } r_i^-(\lambda) = 0\}$$

and, if p is an odd prime number,

(6.8) $$\operatorname{Bip}_{p-\mathrm{reg}}(n) = \{\lambda \in \operatorname{Bip}(n) \mid \forall i \geqslant 1,\ r_i^+(\lambda) \leqslant p-1 \text{ and } r_i^-(\lambda) \leqslant p-1\}.$$

Now, recall from §1.E that the order $o(\lambda)$ of cox_λ is equal to the lowest common multiple of $(2\lambda_1^+, \dots, 2\lambda_r^+, \lambda_1^-, \dots, \lambda_s^-)$, where $\lambda^+ = (\lambda_1^+, \dots, \lambda_r^+)$ and $(\lambda_1^-, \dots, \lambda_s^-)$. Therefore,

(6.9) $$\operatorname{Bip}_{2'}(n) = \{\lambda \in \operatorname{Bip}(n) \mid \forall i \geqslant 1,\ r_i^+(\lambda) = r_{2i}^-(\lambda) = 0\}$$

and, if p is an odd prime number,

(6.10) $$\mathrm{Bip}_{p'}(n) = \{\lambda \in \mathrm{Bip}(n) \mid \forall i \geqslant 1,\ r_{pi}^{+}(\lambda) = r_{pi}^{-}(\lambda) = 0\}.$$

Now, Corollary 6.5 asserts that

(6.11) $$|\,\mathrm{Bip}_{p-\mathrm{reg}}(n)| = |\,\mathrm{Bip}_{p'}(n)|.$$

This can be proved directly by using the descriptions 6.7, 6.8, 6.9 and 6.10 of both sets and by using the classical argument for proving the analogue of 6.11 for partitions instead of bipartitions.

6.C. Character table. Let us now talk about the character table of $K\Sigma'(W_n)$. By Theorem 6.4, the classes of the elements of the family $(x_{\hat{\lambda}})_{\lambda \in \mathrm{Bip}_{p-\mathrm{reg}}(n)}$ in the semisimple quotient $K\Sigma'(W_n)/\operatorname{Rad}(K\Sigma'(W_n))$ form a K-basis of this last space (recall that $\hat{\lambda}$ has been defined in §1.C: it is a representative of $\boldsymbol{\lambda}^{-1}(\lambda)$). Therefore, to compute an irreducible character of $K\Sigma'(W_n)$, we only need to give the values on $(x_{\hat{\lambda}})_{\lambda \in \mathrm{Bip}_{p-\mathrm{reg}}(n)}$. We call the *character table* of $K\Sigma'(W_n)$ the square matrix $(\pi_{\lambda}^{K}(x_{\hat{\mu}}))_{\lambda \in \mathrm{Bip}_{p'}(n), \mu \in \mathrm{Bip}_{p-\mathrm{reg}}(n)}$. By 1.7, we have :

(6.12) *The character table of $K\Sigma'(W_n)$ is upper triangular*

(for the order $\subset$ on $\mathrm{Bip}(n)$).

The character tables of $\mathbb{Q}\Sigma'(W_2)$ and $\mathbb{Q}\Sigma'(W_3)$ will be given at the end of this paper. If $p > 0$, the character table of $K\Sigma'(W_2)$ and $K\Sigma'(W_3)$ are obtained from the previous ones by reduction modulo p and by deleting the appropriate rows and columns.

7. Loewy length

Recall that the Loewy length of a finite dimensional K-algebra A is the smallest natural number $r \geqslant 1$ such that $(\operatorname{Rad} A)^r = 0$. In this section, we shall use the description of the radical obtained in Theorem 6.4 to compute the Loewy length of $K\Sigma'(W_n)$ (except if $p = 2$). But before doing this, we determine the Loewy length of the algebra $K \operatorname{Irr} W_n$:

Proposition 7.1. *The Loewy length of $K \operatorname{Irr} W_n$ is equal to* $$\begin{cases} 1, & \text{if } p = 0; \\ n+1, & \text{if } p = 2; \\ [n/p] + 1, & \text{if } p > 2. \end{cases}$$

Proof. The result is obvious if $p = 0$ so we may, and we will, assume that $p > 0$. If G is a finite group, we denote by $\ell_p(G, 1)$ the Loewy length of the principal block of $K \operatorname{Irr} G$ (see [B2, §3] for the definition of the principal block of $K \operatorname{Irr} G$: it is the unique block on which the degree map $\deg : K \operatorname{Irr} G \to K$ is non-zero). We denote by $\ell_p(n)$ the Loewy length of $K \operatorname{Irr} W_n$. Then

(1) $$\ell_p(n) \geqslant \ell_p(W_n, 1).$$

If p is odd, then it follows from [B2, Proposition 4.7 (d)] that $\ell_p(W_n, 1) = \ell_p(\mathfrak{S}_n, 1)$. But, by [B2, Corollary 5.8], we have $\ell_p(\mathfrak{S}_n, 1) = [n/p] + 1$. On the other hand, since W_n is isomorphic to a subgroup of $\mathfrak{S}_{2n}$ of odd index, it follows from [B2, Proposition 4.7 (a)] that $\ell_2(W_n, 1) \geqslant \ell_2(\mathfrak{S}_{2n}, 1)$. But, by [B2, Corollary 5.8], we have $\ell_2(\mathfrak{S}_{2n}, 1) = n + 1$. So, by using (1), we have proved that

$$\ell_p(n) \geqslant \begin{cases} n+1, & \text{if } p = 2; \\ [n/p] + 1, & \text{if } p > 2. \end{cases} \tag{2}$$

We shall now prove that these inequalities are actually equalities. For this, we shall need some notation. Recall that the p-rank of a finite group G is the maximal possible rank of an elementary abelian p-subgroup of G. It will be denoted by $\mathrm{rk}_p(G)$. We have

$$\mathrm{rk}_p(\mathfrak{S}_n) = [n/p] \qquad \text{and} \qquad \mathrm{rk}_p(W_n) = \begin{cases} n & \text{if } p = 2, \\ [n/p] & \text{if } p > 2. \end{cases}$$

If $\lambda \in \mathrm{Bip}(n)$, we set

$$\varphi_\lambda = \theta_n^K(x_{\hat{\lambda}}) \qquad \text{and} \qquad \mathrm{rk}_p(\lambda) = \mathrm{rk}_p(\mathcal{W}(\hat{\lambda})).$$

In other words, by 6.6, we have

$$\mathrm{rk}_p(\lambda) = \begin{cases} \displaystyle\sum_{i \geqslant 1} \big([r_i^+(\lambda)/2] + r_i^-(\lambda)\big) & \text{if } p = 2, \\ \\ \displaystyle\sum_{i \geqslant 1} \big([r_i^+(\lambda)/p] + [r_i^-(\lambda)/p]\big) & \text{if } p > 2. \end{cases}$$

In particular, $\pi_2(\lambda) \in \{0, 1, 2, \dots n\}$ and, if p is odd, then $\pi_p(\lambda) \in \{0, 1, 2, \dots, [n/p]\}$. Note that $\lambda \in \mathrm{Bip}_{p-\mathrm{reg}}(n)$ if and only if $\pi_p(\lambda) = 0$. Note also that $(\varphi_\lambda)_{\lambda \in \mathrm{Bip}(n)}$ is a K-basis of $K \operatorname{Irr} W_n$ (see Theorem B). Now, by (2), it is sufficient to show that, if $i \geqslant 0$, then

$$\big(\mathrm{Rad}(K \operatorname{Irr} W_n)\big)^i \subset \bigoplus_{\mathrm{rk}_p(\lambda) \geqslant i} K\varphi_\lambda. \tag{3}$$

So let us now prove (3). Let $\mathcal{I}_i = \bigoplus_{\mathrm{rk}_p(\lambda) \geqslant i} K\chi_\lambda$. We denote by $\mathcal{I}_i\mathcal{I}_j$ the space of K-linear combinations of elements of the form xy, where $x \in \mathcal{I}_i$ and $y \in \mathcal{I}_j$. Then

$$\mathcal{I}_i\mathcal{I}_j \subset \mathcal{I}_{i+j}. \tag{4}$$

Proof of (4). We proceed as in [B1, proof of (♣)]. For simplification, we set $\mathcal{N}_C = N_{W_n}(W_C)$ for every $C \in \mathrm{Comp}(n)$. We have, for λ, $\mu \in \mathrm{Bip}(n)$,

$$\varphi_\lambda \varphi_\mu = \sum_{d \in X_{\hat{\lambda}\hat{\mu}}} \varphi_{\boldsymbol{\lambda}(\hat{\lambda} \cap {}^d\hat{\mu})}.$$

The group $\mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu})$ acts on $X_{\hat{\lambda}\hat{\mu}}$ ($\mathcal{W}(\hat{\lambda})$ acts by left multiplication while $\mathcal{W}(\hat{\mu})$ acts by right multiplication). If $d \in X_{\hat{\lambda}\hat{\mu}}$ and $(x, y) \in \mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu})$, then the groups $W_{\hat{\lambda}} \cap {}^dW_{\hat{\mu}}$ and $W_{\hat{\lambda}} \cap {}^{xdy^{-1}}W_{\hat{\mu}} = {}^x(W_{\hat{\lambda}} \cap {}^dW_{\hat{\mu}})$ are conjugate. In other words,

$$\boldsymbol{\lambda}(\hat{\lambda} \cap {}^d\hat{\mu}) = \boldsymbol{\lambda}(\hat{\lambda} \cap {}^{xdy^{-1}}\hat{\mu}).$$

So, if $X'_{\hat{\lambda}\hat{\mu}}$ denotes a set of representatives of $(\mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu}))$-orbits in $X_{\hat{\lambda}\hat{\mu}}$, then

$$\varphi_\lambda \varphi_\mu = \sum_{d \in X'_{\hat{\lambda}\hat{\mu}}} n_{\lambda,\mu,d} \varphi_{\boldsymbol{\lambda}(\hat{\lambda} \cap {}^d\hat{\mu})},$$

where $n_{\lambda,\mu,d}$ denotes the cardinality of the orbit of d. So it is sufficient to show that, if p does not divide $n_{\lambda,\mu,d}$, then $\mathrm{rk}_p(\boldsymbol{\lambda}(\hat{\lambda} \cap {}^d\hat{\mu})) \geqslant \mathrm{rk}_p(\lambda) + \mathrm{rk}_p(\mu)$.

So let $d \in X'_{\hat{\lambda}\hat{\mu}}$ be such that p does not divide $n_{\lambda,\mu,d}$. Let

$$\begin{array}{rccc} \Delta_d : & \mathcal{N}_{\hat{\lambda}} \cap {}^d\mathcal{N}_{\hat{\mu}} & \longrightarrow & \mathcal{N}_{\hat{\lambda}} \times \mathcal{N}_{\hat{\mu}} \\ & w & \longmapsto & (w, d^{-1}wd). \end{array}$$

Let $\tilde{\Delta}_d : \mathcal{N}_{\hat{\lambda}} \cap {}^d\mathcal{N}_{\hat{\mu}} \to \mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu})$ denote the composition of Δ_d with the canonical projection. Then $\tilde{\Delta}_d$ induces an injective morphism $\bar{\Delta}_d : \mathcal{W}(\lambda, \mu, d) \to \mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\mu)$, where $\mathcal{W}(\lambda, \mu, d) = (\mathcal{N}_{\hat{\lambda}} \cap {}^d\mathcal{N}_{\hat{\mu}})/W_{\hat{\lambda} \cap {}^d\hat{\mu}}$. Then it is easily checked that $\bar{\Delta}_d(\mathcal{W}(\lambda, \mu, d))$ is the stabilizer of d in $\mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu})$. In particular,

$$n_{\lambda,\mu,d} = \frac{|\mathcal{W}(\hat{\lambda})|.|\mathcal{W}(\hat{\mu})|}{|\mathcal{W}(\lambda, \mu, d)|}.$$

So, since p does not divide $n_{\lambda,\mu,d}$, this means that, if P a Sylow p-subgroup of $\mathcal{W}(\lambda, \mu, d)$, then $\bar{\Delta}_d(P)$ is a Sylow p-subgroup of $\mathcal{W}(\hat{\lambda}) \times \mathcal{W}(\hat{\mu})$. In particular, $\mathrm{rk}_p(\mathcal{W}(\lambda, \mu, d)) \geqslant \mathrm{rk}_p(\lambda) + \mathrm{rk}_p(\mu)$. Now, $\mathcal{W}(\lambda, \mu, d)$ is a subgroup of $\mathcal{W}(\hat{\lambda} \cap {}^d\hat{\mu})$. So $\mathrm{rk}_p(\boldsymbol{\lambda}(\hat{\lambda} \cap {}^d\hat{\mu})) \geqslant \mathrm{rk}_p(\lambda) + \mathrm{rk}_p(\mu)$, as desired. □

By (4), $\mathcal{I}_i$ is an ideal of $K \operatorname{Irr} W_n = \mathcal{I}_0$. Moreover, again by (4), $\mathcal{I}_1$ consists of nilpotent elements. So $\mathcal{I}_1 \subset \mathrm{Rad}(K \operatorname{Irr} W_n)$. On the other hand, $\dim_K \mathcal{I}_1 = |\operatorname{Bip}(n)| - |\operatorname{Bip}_{p-\mathrm{reg}}(n)|$, so $\dim_K \mathcal{I}_1 = \dim_K\bigl(\mathrm{Rad}(K \operatorname{Irr} W_n)\bigr)$ (see [B2, Corollary 2.16]). So $\mathcal{I}_1 = \mathrm{Rad}(K \operatorname{Irr} W_n)$. But, by (4), $\mathcal{I}_1^i \subset \mathcal{I}_i$. This shows (3), so the proof of the proposition is complete. □

We are now ready to prove the main theorem of this section (compare with [BP, §5.E]):

Theorem 7.2. *If $p \neq 2$, then the Loewy length of $K\Sigma'(W_n)$ is n. If $p = 2$, then this Loewy length lies in $\{n, n+1, \dots, 2n-1\}$.*

Proof. Let $l_p(n)$ denote the Loewy length of $K\Sigma'(W_n)$. If $n = 1$, then the result of the Theorem is easily checked. So we may, and we will, assume that $n \geqslant 2$. The proof will proceed in two steps.

• *First step: upper bound.* We use here the notation of Example 2.10. Let us first prove the following result: if $k \geqslant 0$ and if $x \in \mathrm{Rad}\, K\Sigma'(W_n)$, then:

(1) $$xK\Sigma'_{\mathcal{F}_k^-(n)}(W_n) \subset K\Sigma'_{\mathcal{F}_{k+1}^-(n)}(W_n);$$

(2) *If $p \neq 2$, then $xK\Sigma'_{\mathcal{F}_k(n)}(W_n) \subset K\Sigma'_{\mathcal{F}_{k+1}(n)}(W_n)$.*

Proof of (1) and (2). Let A_n denote the algebra $K\Sigma'(W_n)/K\Sigma'_{\mathcal{F}_1^-(n)}(W_n)$. Recall that $K\Sigma'_{\mathcal{F}_1^-(n)}(W_n)$ is a two-sided ideal of $K\Sigma'(W_n)$ (see Example 2.10). Then $A_n \simeq K[T]/(T(T-2))$, where T is an indeterminate. Indeed, A_n has dimension 2 and is generated by the image t_n of $x_{(-n)}$ and it is easily checked that $t_n^2 = 2t_n$ (this follows for instance from Proposition C (c), from Proposition 2.3, from Lemma 1.10 (c) and from the fact that $|N_{W_n}(\mathfrak{S}_n)/\mathfrak{S}_n| = |\mathcal{W}(-n)| = 2$). In particular, if $p \neq 2$, then $A_n \simeq K \times K$ is split semisimple.

Now, let $D \in \mathrm{Comp}(n)$. Write $D = (d_1, \dots, d_r)$ and let $a = \mathrm{Res}_{D^+}(x)$. Then

$$xx_D = x_{D^+} a x_D^{D^+}.$$

Since Res_{D^+} is a morphism of algebras, a is a nilpotent element of the algebra $\Sigma'(W_{D^+}) \simeq \Sigma'(W_{|d_1|}) \otimes \cdots \otimes \Sigma'(W_{|d_r|})$. In particular, its image $\bar{a}$ in $A_{D^+} = A_{|d_1|} \otimes \cdots \otimes A_{|d_r|}$ is also nilpotent. So, if $D \in \mathcal{F}_k^-(n)$ (respectively if $D \in \mathcal{F}_k(n)$ and $p \neq 2$) then the above description of A_n shows that $xx_D \in K\Sigma'_{\mathcal{F}_{k+1}^-(n)}(W_n)$ (respectively $xx_D \in K\Sigma'_{\mathcal{F}_{k+1}(n)}(W_n)$). $\square$

Since $\mathcal{F}_n(n) = \mathcal{F}_{2n}^-(n) = \varnothing$, then the statement (1) above shows that $l_2(n) \leqslant 2n$ and the statement (2) shows that, if $p \neq 2$, then $l_p(n) \leqslant n$. We shall show now that, if $n \geqslant 2$, then $l_2(n) \leqslant 2n-1$. So let $a_1, \dots, a_{2n-1} \in \mathrm{Rad}\, K\Sigma'(W_n)$. Then, by (a), we have

$$a_1 \dots a_{2n-1} \in K\Sigma'_{\mathcal{F}_{2n}(n)}(W_n) = Kx_{(-1,-1,\dots,-1)}.$$

Let $\lambda \in K$ be such that $a_1 \dots a_{2n-1} = \lambda x_{(-1,-1,\dots,-1)}$. Then $\theta_n^K(a_1 \dots a_{2n-1}) = \lambda\chi_n$, where χ_n is the regular character of W_n. But, since $\theta_n^K(a_i)$ belongs to the radical of the K-algebra $K\,\mathrm{Irr}\, W_n$, since this algebra has Loewy length $\leqslant n+1$ (see Proposition 7.1) and since $n+1 \leqslant 2n-1$ (because $n \geqslant 2$), we get that $\lambda = 0$, as desired. So we have proved the following results:

(3) *If $n \geqslant 2$, then $l_2(n) \leqslant 2n-1$.*

(4) *If $p \neq 2$, then $l_p(n) \leqslant n$.*

• *Second step: lower bound.* Let $a = x_{(n-1,-1)} - x_{(-1,n-1)}$. Then $a \in \mathrm{Ker}\,\theta_n^K$. If $1 \leqslant i \leqslant n$, let C_i denote the signed compositions $(1, \dots, 1, -1, 1, \dots, 1)$ of n, where the -1 term is in position i. Then:

$$a^{n-1} = \sum_{i=1}^{n} (-1)^i \binom{n-1}{i-1} x_{C_i}. \tag{5}$$

Proof of (5). If $1 \leqslant j \leqslant n$ and if $1 \leqslant i \leqslant n+1-j$, we denote by $C_{i,j}$ the signed composition $(-1,\dots,-1,j,-1,\dots,-1)$, where j appears in the i-th position (for instance, $C_{i,1} = C_i$). For simplification, we set $s_{i,j} = x_{C_{i,j}}$. We have in particular $a = s_{1,n-1} - s_{2,n-1}$. We want to show by induction on $k \in \{1,2,\dots,n-1\}$ that

$$(5^+) \qquad a^k = \sum_{i=1}^{k+1} (-1)^i \binom{k}{i-1} s_{i,n-k}.$$

Note that the formula (5) is obtained by specialising k to $n-1$ in the formula (5^+). For proving (5^+) by induction, it is sufficient to show that

$$s_{1,n-1}s_{i,j} = \alpha_{i,j}s_{i,j} + s_{i,j-1} \quad \text{and} \quad s_{2,n-1}s_{i,j} = \alpha_{i,j}s_{i,j} + s_{i+1,j-1}$$

for some $\alpha_{i,j} \in \mathbb{N}$. The first equality is easily checked using the description of $X_{n-1,-1}$ given in the proof of Proposition 5.7. The second one follows from a similar argument. $\square$

The statement (5) above shows that $l_p(n) \geqslant n$. By (3) and (4), the proof of the Theorem is complete. $\square$

REMARK 7.3 - Keep the notation of the proof of the previous Theorem. It is probable that $l_2(n) = 2n-1$ whenever $n \geqslant 2$ (note that $l_2(1) = 2$, $l_2(2) = 3$, $l_2(3) = 5$, $l_2(4) = 7$ and $l_2(5) = 9$). In fact, it is probable that the element a defined in the above proof lies in $(\operatorname{Rad} \mathbb{F}_2\Sigma'(W_n))^2$ (it has been checked for $n \leqslant 5$): this would imply that $l_2(n) = 2n-1$ for $n \geqslant 2$ (see the statement (5) of the above proof). $\square$

8. PROJECTIVE MODULES, CARTAN MATRIX

8.A. Projective modules. If $\lambda \in \operatorname{Bip}(n)$, we denote by $e_\lambda^{\mathbb{Q}} : W_n \to \mathbb{Q}$ the characteristic function of $\mathcal{C}(\lambda)$. We may, and we will, view it as an element of $\mathbb{Q} \operatorname{Irr} W_n$: we have

$$\textbf{(8.1)} \qquad e_\lambda^{\mathbb{Q}} = \frac{|\mathcal{C}(\lambda)|}{|W_n|} \sum_{\chi \in \operatorname{Irr} W_n} \chi(\operatorname{cox}_\lambda)\chi.$$

Then $(e_\lambda^{\mathbb{Q}})_{\lambda \in \operatorname{Bip}(n)}$ is a family of orthogonal primitive idempotents of $\mathbb{Q} \operatorname{Irr} W_n$ such that $\sum_{\lambda \in \operatorname{Bip}(n)} e_\lambda^{\mathbb{Q}} = 1_n$. Since the morphism $\theta_n^{\mathbb{Q}}$ is surjective, there exists [T, Theorem 3.1 (f)] a family $(E_\lambda^{\mathbb{Q}})_{\lambda \in \operatorname{Bip}(n)}$ of primitive idempotents of $\mathbb{Q}\Sigma'(W_n)$ such that

(1) $\forall\, \lambda \in \operatorname{Bip}(n)$, $\theta_n^{\mathbb{Q}}(E_\lambda^{\mathbb{Q}}) = e_\lambda^{\mathbb{Q}}$.
(2) $\forall\, \lambda, \mu \in \operatorname{Bip}(n)$, $E_\lambda^{\mathbb{Q}} E_\mu^{\mathbb{Q}} = E_\mu^{\mathbb{Q}} E_\lambda^{\mathbb{Q}} = \delta_{\lambda\mu} E_\lambda^{\mathbb{Q}}$.
(3) $\sum_{\lambda \in \operatorname{Bip}(n)} E_\lambda^{\mathbb{Q}} = 1$.

Let $\mathcal{P}_\lambda^{\mathbb{Q}} = \mathbb{Q}\Sigma'(W_n)E_\lambda^{\mathbb{Q}}$. It is an indecomposable projective $\mathbb{Q}\Sigma'(W_n)$-module: this is the projective cover of $\mathcal{D}_\lambda^{\mathbb{Q}}$. Moreover,

$$\textbf{(8.2)} \qquad \bigoplus_{\lambda \in \operatorname{Bip}(n)} \mathcal{P}_\lambda^{\mathbb{Q}} = \mathbb{Q}\Sigma'(W_n).$$

If $p = 0$, then $\mathbb{Q} \subset K$ and we set $E_\lambda^K = E_\lambda^{\mathbb{Q}}$. Note that, if $p = 0$, then $(E_\lambda^K)_{\lambda \in \mathrm{Bip}(n)}$ is still a family of orthogonal primitive idempotents of $K\Sigma'(W_n)$ (since $\mathbb{Q}\Sigma'(W_n)$ is split).

EXAMPLE 8.3 - We keep the notation introduced in Example 4.7. If $I \subset I_n^+$, then

$$\boldsymbol{\lambda}(\mathbf{C}(I)) = ((\underbrace{1,1,\dots,1}_{|I| \text{ times}}),(\underbrace{1,1,\dots,1}_{n-|I| \text{ times}})).$$

Then

(8.4) *the idempotent* $x'_{\mathbf{C}(I)}/(2^{n-|I|}|\mathfrak{S}_n(I)|)$ *is conjugate to* $E^{\mathbb{Q}}_{\boldsymbol{\lambda}(\mathbf{C}(I))}$.

Let us prove this result. Since $x'_{\mathbf{C}(I)}/(2^{n-|I|}|\mathfrak{S}_n(I)|)$ is an idempotent (see 4.9), it is sufficient to show that $\theta_n^{\mathbb{Q}}(x'_{\mathbf{C}(I)}/(2^{n-|I|}|\mathfrak{S}_n(I)|)) = \theta_n^{\mathbb{Q}}(E^{\mathbb{Q}}_{\boldsymbol{\lambda}(\mathbf{C}(I))})$ (see [T, Theorem 3.1 (e)]). In other words, it is sufficient to show that $\theta_n^{\mathbb{Q}}(x'_{\mathbf{C}(I)})$ is a multiple of $e^{\mathbb{Q}}_{\boldsymbol{\lambda}(\mathbf{C}(I))}$.

If $J \subset I_n^+$, let $\mathfrak{T}_J$ denote the subgroup of $\mathfrak{T}_n$ generated by $(t_i)_{i\in J}$ and let $t_J = \prod_{i\in J} t_i$. Then $t_J \in \mathcal{C}(\boldsymbol{\lambda}(\mathbf{C}(J)))$ and

$$\theta_n^{\mathbb{Q}}(x'_{\mathbf{C}(I)}) = \mathrm{Ind}_{\mathfrak{T}_I}^{W_n} f_I, \quad \text{where} \quad f_I = \sum_{J\subset I}\Big(-\frac{1}{2}\Big)^{|I|-|J|} 1_J.$$

Here, 1_J denotes the trivial character of $\mathfrak{T}_J$. It is now sufficient to show that f_I is the characteristic function of $\{t_I\}$ in $\mathfrak{T}_I$: since $\mathfrak{T}_I$ is an elementary abelian 2-group, this is easily reduced, by direct products, to the case where $|I| = 1$ (for which it is obvious). □

Let us now assume that $p > 0$. For each $\lambda \in \mathrm{Bip}_{p'}(n)$, we denote by $\mathcal{C}_{p'}(\lambda)$ the set of elements w in W_n such that $w_{p'}$ belongs to $\mathcal{C}(\lambda)$. It is a union of conjugacy classes of W_n. We set

$$e^{\mathbb{Q}}_{\lambda,p'} = \sum_{\substack{\mu\in\mathrm{Bip}(n)\\ \mu_{p'}=\mu}} e^{\mathbb{Q}}_\mu.$$

This is the characteristic function of $\mathcal{C}_{p'}(\lambda)$. It is an idempotent of $\mathbb{Q}\,\mathrm{Irr}\,W_n$ and, by [B2, Corollary 2.21], it is a primitive idempotent of $\mathbb{Z}_{(p)}\,\mathrm{Irr}\,W_n$. We denote by e_λ^K its image in $K\,\mathrm{Irr}\,W_n$: it is still a primitive idempotent of $K\,\mathrm{Irr}\,W_n$. Then $(e_\lambda^K)_{\lambda\in\mathrm{Bip}_{p'}(n)}$ is a family of orthogonal primitive idempotents of $K\,\mathrm{Irr}\,W_n$ such that $\sum_{\lambda\in\mathrm{Bip}_{p'}(n)} e_\lambda^K = 1_n$. Since the morphism θ_n^K is surjective, there exists [T, Theorem 3.1 (f)] a family $(E_\lambda^K)_{\lambda\in\mathrm{Bip}_{p'}(n)}$ of primitive idempotents of $K\Sigma'(W_n)$ such that

(1) $\forall\ \lambda \in \mathrm{Bip}_{p'}(n)$, $\theta_n^K(E_\lambda^K) = e_\lambda^K$.
(2) $\forall\ \lambda,\mu \in \mathrm{Bip}_{p'}(n)$, $E_\lambda^K E_\mu^K = E_\mu^K E_\lambda^K = \delta_{\lambda\mu} E_\lambda^K$.
(3) $\sum_{\lambda\in\mathrm{Bip}(n)} E_\lambda^K = 1$.

Let $\mathcal{P}_\lambda^K = K\Sigma'(W_n)E_\lambda^K$. It is an indecomposable projective $K\Sigma'(W_n)$-module: this is the projective cover of $\mathcal{D}_\lambda^K$. Moreover,

(8.5)
$$\bigoplus_{\lambda\in\mathrm{Bip}_{p'}(n)} \mathcal{P}_\lambda^K = K\Sigma'(W_n).$$

We conclude this section by a useful remark on the idempotents $E_\lambda^{\mathbb{Q}}$: this will be used for proving the unitriangularity of the Cartan matrix of $\mathbb{Q}\Sigma'(W_n)$.

Proposition 8.6. *Let $D \in \mathrm{Comp}(n)$, let $\lambda = \boldsymbol{\lambda}(D)$ and let D' be a semi-positive signed composition of n such that $D \subset D'$. Then there exists a primitive idempotent E of $\mathbb{Q}\Sigma'(W_n)$ satisfying the following two conditions:*

(a) $\theta_n^{\mathbb{Q}}(E) = e_\lambda^{\mathbb{Q}}$.

(b) $E \in \mathbb{Q}\Sigma'(W_n)x_{D'}$.

In particular, E is conjugate to $E_\lambda^{\mathbb{Q}}$.

REMARK 8.7 - In the above Proposition, one can choose $D' = D^+$. □

Proof. For simplification, let $\mathcal{K} = \mathrm{Ker}(\mathrm{Res}_{D'})$ and $\mathcal{I} = \mathbb{Q}\Sigma'(W_n)x_{D'}$. By Proposition 5.4 (b), we have

$$\mathbb{Q}\Sigma'(W_n) = \mathcal{K} \oplus \mathcal{I}.$$

In particular, the restriction of $\mathrm{Res}_{D'}$ to the left ideal $\mathcal{I}$ is injective. Moreover, as a direct consequence of the hypothesis, we get that $\mathrm{Res}_{W_{D'}}^{W_n} e_\lambda^{\mathbb{Q}} \neq 0$, so in particular $\mathrm{Res}_{D'} E_\lambda^{\mathbb{Q}} \neq 0$ (see also Proposition 5.1 (d)). Let us write $E_\lambda^{\mathbb{Q}} = A + E$, with $A \in \mathcal{K}$ and $E \in \mathcal{I}$. Then $E^2 - E \in \mathcal{I}$ and $\mathrm{Res}_{D'}(E^2 - E) = \mathrm{Res}_{D'}((E_\lambda^{\mathbb{Q}})^2 - E_\lambda^{\mathbb{Q}}) = 0$. Therefore, $E^2 = E$. Moreover, $AE \in \mathcal{K} \cap \mathcal{I}$, so $AE = 0$. In other words, $E_\lambda^{\mathbb{Q}}E = E^2 = E$. This shows in particular that

$$(*) \qquad \dim_{\mathbb{Q}} \mathbb{Q}\Sigma'(W_n)E = \dim_{\mathbb{Q}} \mathbb{Q}\Sigma'(W_n)E_\lambda^{\mathbb{Q}}E \leqslant \dim_{\mathbb{Q}} \mathbb{Q}\Sigma'(W_n)E_\lambda^{\mathbb{Q}}.$$

Now, $\mathrm{Res}_{D'}(E_\lambda^{\mathbb{Q}}) = \mathrm{Res}_{D'}(E)$. Since $E_\lambda^{\mathbb{Q}}$ is primitive, this implies that $E = \mathcal{E}_\lambda + F$, where $\mathcal{E}_\lambda$ and F are orthogonal idempotent and $\mathcal{E}_\lambda$ is conjugate to $E_\lambda^{\mathbb{Q}}$ (see [T, Theorem 3.2 (c)]). But, by $(*)$, we get that $F = 0$, so that $\theta_n^{\mathbb{Q}}(E) = e_\lambda^{\mathbb{Q}}$. This shows the proposition. □

8.B. About the structure of KW_n as a left $K\Sigma'(W_n)$. The next result is the analogue of [BBHT, Theorem 7.15]:

Proposition 8.8. *If $p = 0$ and if $\lambda \in \mathrm{Bip}(n)$, then* $\dim_K KW_nE_\lambda^K = |\mathcal{C}(\lambda)|$.

Proof. We may assume that $K = \mathbb{Q}$. Let $\mathcal{T}_n : \mathbb{Q}W_n \to \mathbb{Q}$ denote the unique linear map such that $\mathcal{T}_n(1) = 1$ and $\mathcal{T}_n(w) = 0$ for every $w \in W_n$ which is different from 1. Then $\mathcal{T}_n$ is the canonical symmetrizing form on $\mathbb{Q}W_n$. Now, if $x \in \mathbb{Q}W_n$, then the trace of the multiplication by x on $\mathbb{Q}W_n$ (on the left or on the right) is equal to $|W_n|\mathcal{T}_n(x)$. Therefore, since $E_\lambda^{\mathbb{Q}}$ is an idempotent, $\dim_{\mathbb{Q}} \mathbb{Q}W_nE_\lambda^{\mathbb{Q}} = |W_n|\mathcal{T}_n(E_\lambda^{\mathbb{Q}})$. But, by [BH, Proposition 3.8], we have

$$\mathcal{T}_n(E_\lambda^{\mathbb{Q}}) = \langle \theta_n^{\mathbb{Q}}(E_\lambda^{\mathbb{Q}}), \theta_n^{\mathbb{Q}}(1)\rangle_{W_n} = \langle e_\lambda^{\mathbb{Q}}, 1_n\rangle_{W_n} = \frac{|\mathcal{C}(\lambda)|}{|W_n|},$$

as expected. □

In the same spirit, we have the following result:

Proposition 8.9. *The character of the $\mathbb{Q}\Sigma'(W_n)$-module $\mathbb{Q}W_n$ is* $\displaystyle\sum_{\lambda \in \mathrm{Bip}(n)} |\mathcal{C}(\lambda)|\pi_\lambda^{\mathbb{Q}}$.

Proof. If $C \in \mathrm{Comp}(n)$, the trace of x_C in its left action on $\mathbb{Q}W_n$ is equal to $|W_n|\mathcal{T}_n(x_C) = |W_n|$. On the other hand,

$$\begin{aligned}\sum_{\lambda \in \mathrm{Bip}(n)} |\mathcal{C}(\lambda)|\pi_\lambda^{\mathbb{Q}}(x_C) &= \sum_{\lambda \in \mathrm{Bip}(n)} |\mathcal{C}(\lambda)|\theta_n^{\mathbb{Q}}(x_C)(\mathrm{cox}_\lambda) \\ &= \sum_{\lambda \in \mathrm{Bip}(n)} |W_n|\langle \theta_n^{\mathbb{Q}}(x_C), e_\lambda^{\mathbb{Q}}\rangle_{W_n} \\ &= |W_n|\langle \theta_n^{\mathbb{Q}}(x_C), 1_n\rangle_{W_n} \\ &= |W_n|,\end{aligned}$$

as desired. □

Proposition 8.10. *If $p > 0$ and if $\lambda \in \mathrm{Bip}_{p'}(n)$, then* $\dim_K KW_nE_\lambda^K = |\mathcal{C}_{p'}(\lambda)|$.

Proof. We may, and we will, assume that $K = \mathbb{F}_p$. The idempotent $E_\lambda^{\mathbb{F}_p}$ can be lifted to an idempotent $E_\lambda^{\mathbb{Z}_p}$ of $\mathbb{Z}_p\Sigma'(W_n)$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers [T, Theorem 3.2 (b)]. It is sufficient to show that

$$(?) \qquad \dim_{\mathbb{Q}_p} \mathbb{Q}_p\Sigma'(W_n)E_\lambda^{\mathbb{Z}_p} = |\mathcal{C}_{p'}(\lambda)|.$$

Now, $\theta_n(E_\lambda^{\mathbb{Z}_p})$ is an idempotent of $\mathbb{Z}_p \operatorname{Irr} W_n$ that lifts e_λ^K. Therefore, $\theta_n(E_\lambda^{\mathbb{Z}_p}) = e_{\lambda,p'}^{\mathbb{Q}}$ by the unicity of liftings in commutative algebras [T, Theorem 3.2 (d)]. Therefore, $E_\lambda^{\mathbb{Z}_p}$ is conjugate to the idempotent $\sum_{\mu \in \mathrm{Bip}(n), \mu_{p'}=\lambda} E_\mu^{\mathbb{Q}}$ (see [T, Theorem 3.2 (d)]). So the result follows from Proposition 8.8. □

8.C. Cartan matrix. We return to the general situation, namely we assume that K is a field of characteristic $p \geqslant 0$. We denote by $\mathrm{Cartan}(K\Sigma'(W_n))$ the Cartan matrix of $K\Sigma'(W_n)$. It is the square matrix $([\mathcal{P}_\lambda^K : \mathcal{D}_\mu^K])_{\lambda,\mu \in \mathrm{Bip}_{p'}(n)}$, where $[\mathcal{P}_\lambda^K : \mathcal{D}_\mu^K]$ denotes the multiplicity of $\mathcal{D}_\mu^K$ as a chief factor in a Jordan-Hölder series of $\mathcal{P}_\lambda^K$. Recall that

$$\textbf{(8.11)} \qquad [\mathcal{P}_\lambda^K : \mathcal{D}_\mu^K] = \dim_K \mathrm{Hom}_{K\Sigma'(W_n)}(\mathcal{P}_\mu^K, \mathcal{P}_\lambda^K)$$

and that we have a canonical isomorphism of vector spaces

$$\textbf{(8.12)} \qquad \mathrm{Hom}_{K\Sigma'(W_n)}(\mathcal{P}_\mu^K, \mathcal{P}_\lambda^K) \simeq E_\mu^K K\Sigma'(W_n)E_\lambda^K.$$

Moreover, the isomorphism 8.12 is an isomorphism of algebras whenever $\lambda = \mu$.

Let $D_n^K = (\delta_{\lambda_{p'},\mu})_{\lambda \in \mathrm{Bip}(n), \mu \in \mathrm{Bip}_{p'}(n)}$, where $\delta_{!,?}$ is the Kronecker symbol. If p does not divide the order of $|W_n|$, this is just the identity matrix. In general, it may be seen as the decomposition matrix from $\mathbb{Q}\Sigma'(W_n)$ to $K\Sigma'(W_n)$ (see [GP, §7.4] for the general definition of a decomposition matrix). The next lemma reduces the computation of $\mathrm{Cartan}(K\Sigma'(W_n))$ to the computation of $\mathrm{Cartan}(\mathbb{Q}\Sigma'(W_n))$ by making use of the decomposition matrix D_n^K (see [APVW, Theorem 8] for the analogue of the next result for Solomon descent algebras).

Lemma 8.13. *We have* $\mathrm{Cartan}(K\Sigma'(W_n)) = {}^tD_n^K \mathrm{Cartan}(\mathbb{Q}\Sigma'(W_n))D_n^K$.

Proof. This follows from [GR, §2.3]. □

The next result is a first decomposition of the Cartan matrix of $K\Sigma'(W_n)$ into diagonal blocks (whenever $p \neq 2$), according to the action of w_n on simple modules.

Lemma 8.14. *Assume that $p \neq 2$. Let λ, $\mu \in \mathrm{Bip}_{p'}(n)$. If $[\mathcal{P}_\lambda^K : \mathcal{D}_\mu^K] \neq 0$, then*

$$\lg^-(\lambda) \equiv \lg^-(\mu) \mod 2.$$

Proof. Let λ, $\mu \in \mathrm{Bip}_{p'}(n)$ be such that $[\mathcal{P}_\lambda^K : \mathcal{D}_\mu^K] \neq 0$ and let $? \in \{+,-\}$. First, note that $e_n^? \mathcal{P}_\lambda^K = \mathcal{P}_\lambda^K$ if and only if $e_n^? \mathcal{D}_\lambda^K$ because $\mathcal{P}_\lambda^K$ is indecomposable. On the other hand, if $e_n^? \mathcal{P}_\lambda^K = \mathcal{P}_\lambda^K$, then $e_n^? \mathcal{D}_\mu^K = \mathcal{D}_\mu^K$. So the result follows from 4.2. □

The main result of this section is the following:

Theorem 8.15. *The Cartan matrix* $\mathrm{Cartan}(\mathbb{Q}\Sigma'(W_n))$ *is unitriangular. More precisely, if λ and μ are two distinct bipartitions of n, then:*

(a) $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\lambda^{\mathbb{Q}}] = 1$.

(b) *If* $[\mathcal{P}_\lambda^{\mathbb{Q}} : \mathcal{D}_\mu^{\mathbb{Q}}] \neq 0$, *then* $\lg(\mu) > \lg(\lambda)$.

Proof. Let $\mathcal{F} = \{C \in \mathrm{Comp}(n) \mid \lg(C) > \lg(\lambda)\}$. Then $\mathcal{F}$ is saturated so $\mathcal{I} = \mathbb{Q}\Sigma'_{\mathcal{F}}(W_n)$ is a two-sided ideal of $\mathbb{Q}\Sigma'(W_n)$. The theorem follows from the fact that $\mathcal{P}_\lambda^{\mathbb{Q}} \subset \mathbb{Q}E_\lambda^{\mathbb{Q}} + \mathcal{I}$, which is an immediate consequence of the statement (2) in the proof of the Theorem 7.2. □

We shall give at the end of this paper the Cartan matrices of $\mathbb{Q}\Sigma'(W_2)$, $\mathbb{Q}\Sigma'(W_3)$ and $\mathbb{Q}\Sigma'(W_4)$.

Corollary 8.16. *If p does not divide the order of W_n (i.e. if $p = 0$ or $p > \max(2, n)$), then the centre of $K\Sigma'(W_n)$ is split semisimple.*

Proof. Note that $\mathrm{Bip}(n) = \mathrm{Bip}_{p'}(n)$. Let Z be the centre of $K\Sigma'(W_n)$. Then the map $Z \to \mathrm{End}_K(K\Sigma'(W_n))$ sending $z \in Z$ to the left multiplication by z is injective. Moreover, the image is contained in $\oplus_{\lambda \in \mathrm{Bip}_{p'}(n)} \mathrm{End}_{K\Sigma'(W_n)} \mathcal{P}_\lambda^K$. But, by 8.11, by Lemma 8.13 (and the fact that the matrix D_n^K is the identity) and by Theorem 8.15 (a), we have an isomorphism of K-algebras

$$\mathrm{End}_{K\Sigma'(W_n)} \mathcal{P}_\lambda^K \simeq K.$$

So Z is a subalgebra of $K \times \cdots \times K$ ($|\mathrm{Bip}(n)|$ times). The proof of the corollary is complete. □

EXAMPLE 8.17 - If p divides the order of W_n, then the centre of $K\Sigma'(W_n)$ is not semisimple. Indeed, the element $x_{(-1,-1,\dots,-1)}$ is central in $\mathbb{Q}\Sigma'(W_n)$ and $(x_{(-1,-1,\dots,-1)})^2 = |W_n| x_{(-1,-1,\dots,-1)}$. □

Corollary 8.18. *Let Z_n^R denote the centre of $R\Sigma'(W_n)$. If p does not divide the order of W_n, then the natural map $K \otimes_{\mathbb{Z}} Z_n^{\mathbb{Z}} \to Z_n^K$ is an isomorphism of algebras.*

Proof. It is sufficient to show that $\dim_K Z_n^K = \dim_{\mathbb{Q}} Z_n^{\mathbb{Q}}$. But, since Z_n^K is split semisimple (see Corollary 8.16), its dimension is equal to the number of blocks of $K\Sigma'(W_n)$. This number is determined by the Cartan matrix of $K\Sigma'(W_n)$. Since the Cartan matrices of $K\Sigma'(W_n)$ and $\mathbb{Q}\Sigma'(W_n)$ coincide by Lemma 8.13, the result follows. $\square$

The next example shows that the Corollary 8.18 does not hold for any p and any n.

EXAMPLE 8.19 - It would be interesting to determine the centre of $K\Sigma'(W_n)$. Note that this dimension is always $\geqslant 4$. Indeed, if $p \neq 2$, then $x_n = 1$, w_n, $x_\varnothing$ and $x'_{1,1,\dots,1}$ are linearly independent central elements (see 4.10). If $p = 2$, then $x'_{1,1,\dots,1}$ must me replaced by the image of $2^{n-1}x'_{1,1,\dots,1} - x_\varnothing/2 \in \mathbb{Z}\Sigma'(W_n)$ in $K\Sigma'(W_n)$.

The next table, obtained using `CHEVIE` [Chevie], provides the dimension of this centre for $n \leqslant 5$: it depends on the characteristic p of K.

$n\backslash p$	0	2	$\geqslant 3$
1	2	2	2
2	4	4	4
3	4	4	4
4	5	6	5
5	4	4	4

Note that the case $p > n$ has been handled by using the Lemma 8.18. It would also be interesting to determine for which pairs (p, n) does the Lemma 8.18 hold. For instance, does it hold if p is odd? $\square$

We conclude this section by proving that the Cartan matrix of $\mathbb{Q}\Sigma'(W_n)$ is a submatrix of the Cartan matrix of $\mathbb{Q}\Sigma'(W_{n+1})$. We identify $\mathrm{Bip}(n, -1)$ with $\mathrm{Bip}(n)$ and the map $\tau_{(n,-1)} : \mathrm{Bip}(n) \to \mathrm{Bip}(n+1)$ defined in §5.A will be denoted simply by τ_n. Then:

Theorem 8.20. *Let λ and μ be two bipartitions of n. Then*

$$[\mathcal{P}^{\mathbb{Q}}_{\tau_n(\lambda)} : \mathcal{D}^{\mathbb{Q}}_{\tau_n(\mu)}] = [\mathcal{P}^{\mathbb{Q}}_\lambda : \mathcal{D}^{\mathbb{Q}}_\mu].$$

Proof. By Proposition 8.6, we may, and we will, assume that $E^{\mathbb{Q}}_{\tau_n(\lambda)}$ and $E^{\mathbb{Q}}_{\tau_n(\mu)}$ belong to $\mathbb{Q}\Sigma'(W_{n+1})x_{n,-1}$. In particular, by Proposition 5.4 (b), $E^{\mathbb{Q}}_{\tau_n(\mu)}\mathbb{Q}\Sigma'(W_{n+1})E^{\mathbb{Q}}_{\tau_n(\lambda)}$ is mapped isomorphically to $\mathcal{E}_\lambda\mathbb{Q}\Sigma'(W_n)\mathcal{E}_\mu$ through the map Res_n^{n+1}, where $\mathcal{E}_\lambda = \mathrm{Res}_n^{n+1}(E^{\mathbb{Q}}_\lambda)$ and $\mathcal{E}_\mu = \mathrm{Res}_n^{n+1}(E^{\mathbb{Q}}_\mu)$. So it remains to show that $\mathcal{E}_\lambda$ and $\mathcal{E}_\mu$ are conjugate to $E^{\mathbb{Q}}_\lambda$ and $E^{\mathbb{Q}}_\mu$ respectively (see 8.11 and 8.12).

Let us prove it for λ (this is sufficient). By Proposition 5.1 (d), we have

$$\theta_n^{\mathbb{Q}}(\mathcal{E}_\lambda) = \operatorname{Res}_{W_n}^{W_{n+1}} e_{\tau_n(\lambda)}^{\mathbb{Q}} = e_\lambda^{\mathbb{Q}}.$$

But, $\mathcal{E}_\lambda$ is a primitive idempotent in the image of $\operatorname{Res}_n^{n+1}$ (see [T, Theorem 3.2 (d)]) and since $\operatorname{Res}_n^{n+1}$ is surjective (see Proposition 5.7), we get that $\mathcal{E}_\lambda$ is a primitive idempotent of $\mathbb{Q}\Sigma'(W_n)$. So $\mathcal{E}_\lambda$ and $E_\lambda^{\mathbb{Q}}$ are conjugate [T, Theorem 3.2 (c)]. □

9. Numerical results

For simplification, a bipartition $((\lambda_1^+,\dots,\lambda_r^+),(\lambda_1^-,\dots,\lambda_s^-))$ will be denoted in a compact way $\lambda_1^+\dots\lambda_r^+;\lambda_1^-\dots\lambda_s^-$. For instance, $31;411$ stands for $((3,1),(4,1,1))$ and $\varnothing;221$ stands for $((),(2,2,1))$. If $i \geqslant 1$, the number $-i$ will be denoted by $\bar{i}$: for instance, the signed composition $(2,-3,-1,1,-2)$ will be denoted by $(2,\bar{3},\bar{1},1,\bar{2})$.

We shall give here the Cartan matrix and the primitive central idempotents of the algebras $\mathbb{Q}\Sigma'(W_n)$ for $n\in\{2,3,4\}$. For $n\in\{2,3\}$, we also give the character table and an example of a family $(E_\lambda^{\mathbb{Q}})_{\lambda\in\operatorname{Bip}(n)}$. Note that they are obtained by lifting the idempotents $(e_\lambda^{\mathbb{Q}})_{\lambda\in\operatorname{Bip}(n)}$ by using `CHEVIE` [Chevie] and the algorithm described in [T, Theorem 3.1 (b) and (f)]. In the next tables, we have replaced zeroes by dots. Note also that, for simplicity, the idempotents will be expressed in the basis $(x'_C)_{C\in\operatorname{Comp}(n)}$ constructed in Section 4.

9.A. The case $n=2$. The character table of $\mathbb{Q}\Sigma'(W_2)$ is:

	x_2	$x_{\bar{2}}$	$x_{1,1}$	$x_{1,\bar{1}}$	$x_{\bar{1},\bar{1}}$
$\pi_{2;\varnothing}^{\mathbb{Q}}$	1	.	.	.	.
$\pi_{\varnothing;2}^{\mathbb{Q}}$	1	2	.	.	.
$\pi_{11;\varnothing}^{\mathbb{Q}}$	1	.	2	.	.
$\pi_{1;1}^{\mathbb{Q}}$	1	.	2	2	.
$\pi_{\varnothing;11}^{\mathbb{Q}}$	1	4	2	4	8

We can take for the family $(E_\lambda^{\mathbb{Q}})_{\lambda\in\operatorname{Bip}(2)}$ the following idempotents:

$$\begin{aligned}
E_{2;\varnothing}^{\mathbb{Q}} &= x'_2 - \frac{1}{2}x'_{1,1} + \frac{1}{8}x'_{\bar{1},\bar{1}}\\
E_{\varnothing;2}^{\mathbb{Q}} &= \frac{1}{2}x'_{\bar{2}} + \frac{1}{4}(x'_{1,\bar{1}} - x'_{\bar{1},1})\\
E_{11;\varnothing}^{\mathbb{Q}} &= \frac{1}{2}x'_{1,1}\\
E_{1;1}^{\mathbb{Q}} &= \frac{1}{4}(x'_{1,\bar{1}} + x'_{\bar{1},1})\\
E_{\varnothing;11}^{\mathbb{Q}} &= \frac{1}{8}x'_{\bar{1},\bar{1}}.
\end{aligned}$$

The Cartan matrix of $\mathbb{Q}\Sigma'(W_2)$ is given by:

	$\mathcal{D}^{\mathbb{Q}}_{2;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;2}$	$\mathcal{D}^{\mathbb{Q}}_{1;1}$	$\mathcal{D}^{\mathbb{Q}}_{11;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;11}$
$\mathcal{P}^{\mathbb{Q}}_{2;\varnothing}$	1	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;2}$	.	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;1}$	.	.	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{11;\varnothing}$	.	.	.	1	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;11}$	.	.	.	.	1

The primitive central idempotents of $\mathbb{Q}\Sigma'(W_2)$ are

$$\begin{aligned}
F_1 &= x'_2 - \frac{1}{2}x'_{1,1} + \frac{1}{8}x'_{\bar{1},\bar{1}} \\
F_2 &= \frac{1}{2}(x'_{\bar{2}} + x'_{1,\bar{1}}) \\
F_3 &= \frac{1}{2}x'_{1,1} \\
F_4 &= \frac{1}{8}x'_{\bar{1},\bar{1}}
\end{aligned}$$

If we denote by A_i the block $\mathbb{Q}\Sigma'(W_2)F_i$, then $A_i \simeq \mathbb{Q}$ if $i \in \{1,3,4\}$ and A_2 is isomorphic to the algebra to upper triangular 2×2-matrices (see [BH, §6]). In particular, $\mathbb{Q}\Sigma'(W_2)$ is hereditary.

For information, we provide the dimensions of the left ideal, right ideal, two-sided ideal generated by x_C (for $C \in \mathrm{Comp}(2)$) and also the dimension of the centralizer of x_C. In this table, A denotes the algebra $\mathbb{Q}\Sigma'(W_2)$.

x	x_2	$x_{\bar{2}}$	$x_{1,1}$	$x_{1,\bar{1}}$	$x_{\bar{1},1}$	$x_{\bar{1},\bar{1}}$
$\dim_{\mathbb{Q}} Ax$	6	3	3	2	2	1
$\dim_{\mathbb{Q}} xA$	6	2	4	3	3	1
$\dim_{\mathbb{Q}} AxA$	6	3	4	3	3	1
$\dim_{\mathbb{Q}} Z_A(x)$	6	5	5	5	5	6

9.B. The case $n = 3$. The character table of $\mathbb{Q}\Sigma'(W_3)$ is:

	x_3	$x_{\bar{3}}$	$x_{2,1}$	$x_{2,\bar{1}}$	$x_{1,\bar{2}}$	$x_{\bar{2},\bar{1}}$	$x_{1,1,1}$	$x_{1,1,\bar{1}}$	$x_{1,\bar{1},\bar{1}}$	$x_{\bar{1},\bar{1},\bar{1}}$
$\pi^{\mathbb{Q}}_{3;\varnothing}$	1	.	.	.	.	.	.	.	.	.
$\pi^{\mathbb{Q}}_{\varnothing;3}$	1	2	.	.	.	.	.	.	.	.
$\pi^{\mathbb{Q}}_{21;\varnothing}$	1	.	1	.	.	.	.	.	.	.
$\pi^{\mathbb{Q}}_{2;1}$	1	.	1	2	.	.	.	.	.	.
$\pi^{\mathbb{Q}}_{1;2}$	1	.	1	.	2	.	.	.	.	.
$\pi^{\mathbb{Q}}_{\varnothing;21}$	1	4	1	2	2	4	.	.	.	.
$\pi^{\mathbb{Q}}_{111;\varnothing}$	1	.	3	.	.	.	6	.	.	.
$\pi^{\mathbb{Q}}_{11;1}$	1	.	3	2	.	.	6	4	.	.
$\pi^{\mathbb{Q}}_{1;11}$	1	.	3	4	4	.	6	8	8	.
$\pi^{\mathbb{Q}}_{\varnothing;111}$	1	8	3	6	12	24	6	12	24	48

We can take for the family $(E^{\mathbb{Q}}_\lambda)_{\lambda\in\mathrm{Bip}(3)}$ the following idempotents:

$$\begin{aligned}
E^{\mathbb{Q}}_{3;\varnothing} &= x'_3 - x'_{1,2} + \frac{1}{4}x'_{\bar{1},\bar{2}} + \frac{1}{3}x'_{1,1,1} - \frac{1}{6}x'_{\bar{1},1,\bar{1}} + \frac{1}{12}(x'_{1,\bar{1},\bar{1}} + x'_{\bar{1},\bar{1},1})\\
E^{\mathbb{Q}}_{\varnothing;3} &= \frac{1}{2}(x'_{\bar{3}} + x'_{2,\bar{1}} - x'_{\bar{1},2}) - \frac{1}{3}x'_{1,1,\bar{1}} + \frac{1}{6}(x'_{1,\bar{1},1} + x'_{\bar{1},1,1,})\\
E^{\mathbb{Q}}_{21;\varnothing} &= x'_{1,2} - \frac{1}{2}x'_{1,1,1} + \frac{1}{8}x'_{1,\bar{1},\bar{1}}\\
E^{\mathbb{Q}}_{2;1} &= \frac{1}{2}x'_{\bar{1},2} - \frac{1}{4}x'_{\bar{1},1,1} + \frac{1}{16}x'_{\bar{1},\bar{1},\bar{1}}\\
E^{\mathbb{Q}}_{1;2} &= \frac{1}{2}x'_{1,\bar{2}} + \frac{1}{4}(x'_{1,1,\bar{1}} - x'_{1,\bar{1},1})\\
E^{\mathbb{Q}}_{\varnothing;21} &= \frac{1}{4}x'_{\bar{2},\bar{1}} + \frac{1}{8}(x'_{1,\bar{1},1} - x'_{\bar{1},1,\bar{1}})\\
E^{\mathbb{Q}}_{111;\varnothing} &= \frac{1}{6}x'_{1,1,1}\\
E^{\mathbb{Q}}_{11;1} &= \frac{1}{12}(x'_{1,1,\bar{1}} + x'_{1,\bar{1},1} + x'_{\bar{1},1,1})\\
E^{\mathbb{Q}}_{1;11} &= -\frac{1}{12}x'_{1,\bar{1},\bar{1}} + \frac{7}{24}x'_{\bar{1},1,\bar{1}} - \frac{1}{12}x'_{\bar{1},\bar{1},1}\\
E^{\mathbb{Q}}_{\varnothing;111} &= \frac{1}{48}x'_{\bar{1},\bar{1},\bar{1}}
\end{aligned}$$

The Cartan matrix of $\mathbb{Q}\Sigma'(W_3)$ is given by

	$\mathcal{D}^{\mathbb{Q}}_{3;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{21;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;21}$	$\mathcal{D}^{\mathbb{Q}}_{1;11}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;3}$	$\mathcal{D}^{\mathbb{Q}}_{2;1}$	$\mathcal{D}^{\mathbb{Q}}_{1;2}$	$\mathcal{D}^{\mathbb{Q}}_{11;1}$	$\mathcal{D}^{\mathbb{Q}}_{111;\varnothing}$	$\mathcal{D}^{\mathbb{Q}}_{\varnothing;111}$
$\mathcal{P}^{\mathbb{Q}}_{3;\varnothing}$	1	1	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{21;\varnothing}$	.	1	.	.	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;21}$	.	.	1	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;11}$	.	.	.	1	.	.	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;3}$	.	.	.	.	1	1	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{2;1}$	.	.	.	.	.	1	.	.	.	.
$\mathcal{P}^{\mathbb{Q}}_{1;2}$	.	.	.	.	.	.	1	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{11;1}$	.	.	.	.	.	.	.	1	.	.
$\mathcal{P}^{\mathbb{Q}}_{111;\varnothing}$	.	.	.	.	.	.	.	.	1	.
$\mathcal{P}^{\mathbb{Q}}_{\varnothing;111}$	.	.	.	.	.	.	.	.	.	1

The primitive central idempotents of $\mathbb{Q}\Sigma'(W_3)$ are

$$\begin{aligned}
F_1 &= x'_3 + \frac{1}{4}(x'_{\bar{2},\bar{1}} + x'_{\bar{1},\bar{2}} + x'_{1,\bar{1},\bar{1}}) - \frac{1}{6}x'_{1,1,1} \\
F_2 &= \frac{1}{2}(x'_{\bar{3}} + x'_{2,\bar{1}} + x'_{1,\bar{2}}) + \frac{5}{48}x'_{\bar{1},\bar{1},\bar{1}} \\
F_3 &= \frac{1}{6}x'_{1,1,1} \\
F_4 &= \frac{1}{48}x'_{\bar{1},\bar{1},\bar{1}}
\end{aligned}$$

For information, we provide the dimensions of the left ideal, right ideal, two-sided ideal generated by x_C (for $C \in \mathrm{Comp}(3)$) and also the dimension of the centralizer of x_C. In these tables, A denotes the algebra $\mathbb{Q}\Sigma'(W_3)$.

x	x_3	$x_{\bar{3}}$	$x_{2,1}$	$x_{1,2}$	$x_{2,\bar{1}}$	$x_{\bar{1},2}$	$x_{\bar{2},1}$	$x_{1,\bar{2}}$	$x_{\bar{2},\bar{1}}$	$x_{\bar{1},\bar{2}}$
$\dim_{\mathbb{Q}} Ax$	18	7	10	10	6	6	6	6	3	3
$\dim_{\mathbb{Q}} xA$	18	4	16	16	11	11	8	8	3	3
$\dim_{\mathbb{Q}} AxA$	18	9	16	16	11	11	10	10	5	5
$\dim_{\mathbb{Q}} Z_A(x)$	18	13	10	10	12	12	13	13	16	16

x	$x_{1,1,1}$	$x_{1,1,\bar{1}}$	$x_{1,\bar{1},1}$	$x_{\bar{1},1,1}$	$x_{1,\bar{1},\bar{1}}$	$x_{\bar{1},1,\bar{1}}$	$x_{\bar{1},\bar{1},1}$	$x_{\bar{1},\bar{1},\bar{1}}$
$\dim_{\mathbb{Q}} Ax$	4	3	3	3	2	2	2	1
$\dim_{\mathbb{Q}} xA$	8	7	7	7	4	4	4	1
$\dim_{\mathbb{Q}} AxA$	8	7	7	7	4	4	4	1
$\dim_{\mathbb{Q}} Z_A(x)$	14	14	14	14	16	16	16	18

9.C. The case $n = 4$. We shall give here only the Cartan matrix and the central idempotents of A. The Cartan matrix is given by

	4 ∅	31 ∅	∅ 31	∅ 22	211 ∅	2 11	1 21	11 11	∅ 4	3 1	1 3	2 2	21 1	11 2	∅ 211	111 1	1 111	22 ∅	1111 ∅	∅ 1111
$\mathcal{P}_{4;\varnothing}^{\mathbb{Q}}$	1	1	1	.	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{31;\varnothing}^{\mathbb{Q}}$	.	1	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{\varnothing;31}^{\mathbb{Q}}$	.	.	1	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{\varnothing;22}^{\mathbb{Q}}$	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{211;\varnothing}^{\mathbb{Q}}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{2;11}^{\mathbb{Q}}$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{1;21}^{\mathbb{Q}}$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{11;11}^{\mathbb{Q}}$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$\mathcal{P}_{\varnothing;4}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	1	1	1	1	2	1	1	1	1	.	.	.
$\mathcal{P}_{3;1}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1	.	1	.	.	.
$\mathcal{P}_{1;3}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	1	.	1	1	.	1	.	.	.	.
$\mathcal{P}_{2;2}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
$\mathcal{P}_{21;1}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$\mathcal{P}_{11;2}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.
$\mathcal{P}_{\varnothing;211}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.
$\mathcal{P}_{111;1}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
$\mathcal{P}_{1;111}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
$\mathcal{P}_{22;\varnothing}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$\mathcal{P}_{1111;\varnothing}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$\mathcal{P}_{\varnothing;1111}^{\mathbb{Q}}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

The primitive central idempotents of $\mathbb{Q}\Sigma'(W_4)$ are

$$\begin{aligned}
F_1 &= x'_4 - \frac{1}{2}x'_{2,2} + \frac{1}{4}(x_{\bar{3},\bar{1}} + x'_{\bar{1},\bar{3}} + x'_{2,1,1} + x'_{1,1,2} + x'_{\bar{2},\bar{2}} + x'_{1,\bar{1},\bar{2}} + x'_{1,\bar{2},\bar{1}}) \\
&\quad + \frac{1}{6}x'_{1,1,1,1} + \frac{3}{16}x'_{2,\bar{1},\bar{1}} - \frac{1}{16}x'_{\bar{1},\bar{1},2} + \frac{1}{32}(x'_{1,1,\bar{1},\bar{1}} + x'_{\bar{1},\bar{1},1,1}) + \frac{5}{96}x'_{\bar{1},\bar{1},\bar{1},\bar{1}} \\
F_2 &= \frac{1}{2}(x'_{\bar{4}} + x'_{3,\bar{1}} + x'_{1,\bar{3}} + x'_{2,\bar{2}}) + \frac{1}{8}(x'_{\bar{2},\bar{1},\bar{1}} + x'_{\bar{1},\bar{2},\bar{1}} + x'_{\bar{1},\bar{1},\bar{2}} + x'_{1,\bar{1},\bar{1},\bar{1}}) \\
F_3 &= \frac{1}{2}x'_{2,2} - \frac{1}{4}(x'_{2,1,1} + x'_{1,1,2}) + \frac{1}{8}x'_{1,1,1,1} + \frac{1}{16}(x'_{2,\bar{1},\bar{1}} + x'_{\bar{1},\bar{1},2}) \\
&\quad - \frac{1}{32}(x'_{1,1,\bar{1},\bar{1}} + x'_{\bar{1},\bar{1},1,1}) + \frac{1}{128}x'_{\bar{1},\bar{1},\bar{1},\bar{1}} \\
F_4 &= \frac{1}{24}x'_{1,1,1,1} \\
F_5 &= \frac{1}{384}x'_{\bar{1},\bar{1},\bar{1},\bar{1}}
\end{aligned}$$

10. QUESTIONS

Let us raise here some questions about the representation theory of the Mantaci-Reutenauer algebra $K\Sigma'(W_n)$:

(1) Determine the centre of $K\Sigma'(W_n)$, or at least its dimension (in characteristic zero, its dimension determines its structure because it is split semisimple by Corollary 8.16).

(2) Compute the Cartan matrix of $\mathbb{Q}\Sigma'(W_n)$. Note that the Theorem 8.20 provides a first induction argument.

(2^+) Determine the Loewy series of the projective indecomposable $K\Sigma'(W_n)$-modules. Determine the Loewy length of $\mathbb{F}_2\Sigma'(W_n)$ (see Remark 7.3: it is probably equal to $2n-1$ if $n \geqslant 2$).

(3) For which values of n is the algebra $\mathbb{Q}\Sigma'(W_n)$ hereditary? It is reasonable to expect that it is hereditary if and only if $n \in \{1, 2, 3\}$. Note that it is not hereditary for $n \in \{4, 5\}$.

(3^+) Compute the path algebra of $\mathbb{Q}\Sigma'(W_n)$.

(4) Is the inclusion (4) in the proof of Proposition 7.1 always an equality? For the analogous statement for the symmetric group, we have an equality [B1, Theorem A].

(5) Is the morphism $\mathrm{Res}_k^n : \mathbb{Z}\Sigma'(W_n) \to \mathbb{Z}\Sigma'(W_k)$ surjective? Compare with Proposition 5.7. Note that the morphism $\mathrm{Res}_{W_k}^{W_n} : \mathbb{Z}\,\mathrm{Irr}\, W_n \to \mathbb{Z}\,\mathrm{Irr}\, W_k$ is surjective.

(6) The Corollary 8.8 suggests, by analogy with the case of the symmetric group, the following question: if $\lambda \in \mathrm{Bip}(n)$, does there exist a linear character ζ_λ of $C_{W_n}(\mathrm{cox}_\lambda)$ such that $\mathbb{C}W_n E_\lambda^{\mathbb{C}}$ affords the character $\mathrm{Ind}_{C_{W_n}(\mathrm{cox}_\lambda)}^{W_n} \zeta_\lambda$? In fact, the answer to this question is negative in general, even for $n = 2$ (take $\lambda = ((2); \varnothing)$). Computations using `CHEVIE` (for $n \leqslant 4$) suggests that the following slight modification of the previous question could have a positive answer: if $\lambda \in \mathrm{Bip}(n)$, does there exist a linear character ζ_λ of $C_{W_n}(\mathrm{cox}_{\lambda^{\mathrm{opp}}})$

such that $\mathbb{C}W_nE_\lambda^{\mathbb{C}}$ affords the character $\mathrm{Ind}_{C_{W_n}(\mathrm{cox}_{\lambda^{\mathrm{opp}}})}^{W_n}\zeta_\lambda$? Here, if $\lambda = (\lambda^+, \lambda^-)$, we have denoted by λ^{opp} the bipartition (λ^-, λ^+).

REMARK - It is readily seen that $|C_{W_n}(w_\lambda)| = |C_{W_n}(w_{\lambda^{\mathrm{opp}}})|$. $\square$

EXAMPLE 10.1 - We keep here the notation of Example 4.7. Let $r \in \{0, 1, 2, \dots, n\}$. Then

$$\mathbf{C}(I_r^+) = (\underbrace{1, 1, \dots, 1}_{r \text{ times}}, \underbrace{-1, -1, \dots, -1}_{n-r \text{ times}})$$

and, if we set $\lambda(r) = \boldsymbol{\lambda}(\mathbf{C}(I_r^+))$, then

$$\lambda(r) = (\underbrace{(1, 1, \dots, 1)}_{r \text{ times}}, \underbrace{(1, 1, \dots, 1)}_{n-r \text{ times}}).$$

We shall prove that the answer to question 6 is positive whenever $\lambda = \lambda(r)$.

Let us make this statement more precise. We may choose for $\mathrm{cox}_{\lambda(r)^{\mathrm{opp}}}$ the element $\mathrm{cox}_{\mathbf{C}(I_r^+)} = t_{r+1}t_{r+2}\dots t_n$. Then

$$C_{W_n}(\mathrm{cox}_{\lambda(r)^{\mathrm{opp}}}) = W_{r,n-r}.$$

Let $\gamma_r \boxtimes 1_{n-r}$ denote the linear character of $W_{r,n-r} \simeq W_r \times W_{n-r}$ which is equal to γ_r on the component W_r and which is trivial on W_{n-r} (recall that the linear character γ_r of W_r has been defined in Example 4.7). Then:

(10.2) *The $\mathbb{Q}W_n$-module $\mathbb{Q}W_nE_{\lambda(r)}$ affords the character* $\mathrm{Ind}_{W_{r,n-r}}^{W_n}(\gamma_r \boxtimes 1_{n-r})$.

Let us prove 10.2. First, note that $\gamma_r \boxtimes 1_{n-r}$ is just the restriction of the map $\gamma_{I_r^+} : W_n \to \{1, -1\}$ to $W_{r,n-r}$ defined in Example 4.7. By Example 8.3, we may take for $E_{\lambda(r)}^{\mathbb{Q}}$ the idempotent $x'_{I_r^+}/(2^{n-r}r!(n-r)!)$. Let $e_{\mathfrak{S}_n} = (1/n!)\sum_{\sigma \in \mathfrak{S}_n} \sigma$ and let

$$E(r) = \frac{1}{|W_{r,n-r}|} \sum_{\sigma \in W_{r,n-r}} (\gamma_r \boxtimes 1_{n-r})(\sigma)\sigma.$$

Then the module $\mathbb{Q}W_nE(r)$ affords the character $\mathrm{Ind}_{W_{r,n-r}}^{W_n}(\gamma_r \boxtimes 1_{n-r})$ and, by Example 8.3, we have $E_{\lambda(r)} = (|W_n|/|W_{r,n-r})e_{\mathfrak{S}_n}E(r)$. Therefore, $\mathbb{Q}W_nE_{\lambda(r)} \subset \mathbb{Q}W_nE(r)$. But $\dim_{\mathbb{Q}} \mathbb{Q}W_nE(r) = |W_n|/|W_{r,n-r}|$ and $\dim_{\mathbb{Q}} \mathbb{Q}W_nE_{\lambda(r)} = |\mathcal{C}(\lambda(r))| = |W_n|/|W_{r,n-r}|$, so we get that $\mathbb{Q}W_nE_{\lambda(r)} = \mathbb{Q}W_nE(r)$. $\square$

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